Problem 4: Eigenvalue solution to the Smoluchowski equation

Consider the 1-dimensional Smoluchowski equation

$$\frac{\partial W(x,t)}{\partial t} = \frac{1}{m\gamma}\frac{\partial}{\partial x}\left[kT\frac{\partial}{\partial x} + \frac{\partial U}{\partial x}\right] W(x,t) = -\frac{\partial}{\partial x} S(x)$$

for a harmonic potential

$$U(x) = \frac{m\omega^2}{2} x^2$$

Show that the probability current can be written as

$$S(x,t) = -\frac{kT}{m\gamma} e^{-U(x)/kT} \left(\frac{\partial}{\partial x} e^{U(x)/kT} W(x,t)\right)$$

and that the Fokker-Planck operator can be written as

$$\mathfrak{L}_{FP} = \frac{1}{m\gamma}\frac{\partial}{\partial x}\left[kT\frac{\partial}{\partial x} + \frac{\partial U}{\partial x}\right] = \frac{kT}{m\gamma}\frac{\partial}{\partial x} e^{-U(x)/kT} \frac{\partial}{\partial x} e^{U(x)/kT}$$

and can be transformed into a hermitian operator by

$$\mathfrak{L} = e^{U(x)/2kT} \mathfrak{L}_{FP} e^{-U(x)/2kT}$$

Solve the Eigenvalue problem

$$\mathfrak{L}\psi_n(x) = \lambda_n \psi_n(x)$$

and use the function $\psi_0(x)$ to construct a special solution $W(x,t) = e^{\lambda_0 t} e^{-U(x)/2kT} \psi_0(x)$