Spectral and metric aspects of the Dolbeault-Dirac spectral triple on quantum SO(5)/(SO(2)xSO(3))

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Dirac operator on quantum flag manifolds

Aim : Providing examples to study

Metric and Spectral Aspects in Noncommutative Geometry

Claim: Quantum flag manifolds belong to the best examples!

Features: Hermitian symmetric spaces $M := G/H$

- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \ [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \ [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$
- $T_e G = \mathfrak{h} \oplus \mathfrak{m} = T_e^{\mathrm{vert}} G \oplus T_e^{\mathrm{hor}} G$

$\Rightarrow$ Get a connection for free!

- Dolbeault-Dirac operator: $\bar{\partial} + \bar{\partial}^*$ on $\Omega^{(0,0)} \oplus \ldots \oplus \Omega^{(0,n)}$
- Spin bundle: $(\Omega^{(0,0)} \oplus \ldots \oplus \Omega^{(0,n)}) \otimes \sqrt{\Omega^{(n,0)}}$

$\Rightarrow$ Get the Clifford multiplication for free!

Spin Geometry of Quantum Flag Manifolds

Best case: Standard Podleś sphere $\mathrm{S}^2_q := \mathrm{SU}_q(2)/\mathrm{U}(1) \cong \mathbb{C}\mathrm{P}^1_q$

Dabrowski/Sitarz: Dirac operator on the standard Podleś quantum sphere, BCP 61, 2002.

- Spin bundle: $\mathcal{S} = \mathcal{O}(\mathrm{SU}_q(2)) \,\square_\pi\, (V_1 \oplus V_{-1}) = (\Omega^{(0,0)} \oplus \Omega^{(0,1)}) \otimes \mathcal{E}_{-1}$
- Dirac operator: $D = \begin{pmatrix} 0 & \partial_E \\ \partial_F & 0 \end{pmatrix}$, E, F invariant vector fields
- even, real structure (Tomita operator), first order condition
- 0^+-summable with eigenvalues $[k+1]_q := \frac{q^k - q^{-k}}{q - q^{-1}}$, $q \in (0,1)$
- Twisted Hochschild cycle: $\sum_i a_i^0 \otimes a_i^1 \otimes a_i^2 \in \mathrm{HH}^{\theta^{-1}}(\mathcal{O}(\mathrm{S}^2_q))$
- Invariant volume form: $\sum_i a_i^0 \mathrm{d}a_i^1 \wedge \mathrm{d}a_j^2 \in \Omega^2(\mathrm{S}^2_q)$
- Orientation: $\sum_i a_i^0 [D, a_i^1][D, a_i^2] = \gamma_q := \mathrm{diag}[q^{-1}, -q]$
- Chern character, (q-)index computations, Poincaré Duality
- ...

Higher Dimensional Examples

A-Series:

- F. D'Andrea, L. Dabrowski, G. Landi, 2008: $\mathbb{C}\mathrm{P}_q^2$
- F. D'Andrea, L. Dabrowski, 2010: $\mathbb{C}\mathrm{P}_q^n$
- R. Ó Buachalla, B. Das, P. Somberg, 2020: Kähler structures on $\mathbb{C}\mathrm{P}_q^n$

B-Series:

- R. Ó Buachalla, F. Díaz, E. Wagner, 2022: $\mathrm{SO}_q(5)/(\mathrm{SO}(2)x\mathrm{SO}_q(3))$

from Bernstein-Gelfand-Gelfand resolution and quantum tangent space

Heckenberger/Kolb: J. Geom. Phys. 57, 2007.

Next "Podleś sphere": Spin Geometry of $\mathbb{C}\mathrm{P}_q^3$ **– An invitation!**

Why irreducible quantum flag manifolds?

- $U_q(\mathfrak{l}) \subset U_q(\mathfrak{g})$ Levi factor, $E_{i_0} \in U_q(\mathfrak{g}) \setminus U_q(\mathfrak{l})$ simple root vector
- $B := \mathcal{O}(G_q)^{\mathrm{inv}(U_q(\mathfrak{l}))} = \{f \in \mathcal{O}(G_q) : X \triangleright f = \varepsilon(X) f \;\; \forall X \in U_q(\mathfrak{l})\}$
- $\{K_1, \dots, K_l, E_1, \dots, E_l\} \setminus \{E_{i_0}\} \subset U_q(\mathfrak{l}), \quad a, b \in B$

$\Rightarrow E_{i_0} \triangleright (ab) = (E_{i_0} \triangleright a)(\cancel{K_{i_0}} \triangleright b) + a(E_{i_0} \triangleright b) = (E_{i_0} \triangleright a) b + a(E_{i_0} \triangleright b)$

$\Rightarrow$ First order derivation!

- Tangent space: $\mathfrak{m} := \mathrm{ad}_q(U_q(\mathfrak{l}))(E_{i_0})$, where $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{l} \oplus \mathfrak{m}^*$

$\Rightarrow \mathrm{ad}_q(E_j)(E_{i_0}) \triangleright b = (E_j E_{i_0} - E_{i_0} E_j)\cancel{K_j^{-1}} \triangleright b = E_j E_{i_0} \triangleright b - E_{i_0} \cancel{E_j \triangleright b} = E_j E_{i_0} \triangleright b$

$\Rightarrow E_j E_{i_0} \triangleright (ab) = (E_j E_{i_0} \triangleright a)(\cancel{K_j K_{i_0}} \triangleright b) + (\cancel{E_j \triangleright a})(K_j E_{i_0} \triangleright b) + (E_{i_0} \triangleright a)(\cancel{E_j K_{i_0} \triangleright b}) + a\, E_j E_{i_0} \triangleright b$

$= (E_j E_{i_0} \triangleright a) b + a(E_j E_{i_0} \triangleright b)$

$\Rightarrow \mathfrak{m}\restriction_B = \mathrm{span}\{E_{i_k} \cdots E_{i_1} E_{i_0} \restriction_B : k = 0, \dots, \dim(\mathfrak{m}) - 1\}$, 1^{st}-order diff. operators

Elliptic complex: Bernstein-Gelfand-Gelfand resolution

Classical flag manifold: $\mathfrak{g}$ simple Lie algebra, $\mathfrak{p} \subset \mathfrak{g}$ of parabolic type

Classical BGG resolution: $X_1 = [Y_1], \ldots, X_k = [Y_k] \in \mathfrak{g}/\mathfrak{p}, \quad Z \in U(\mathfrak{g})$

$$\Rightarrow \delta_k(Z \otimes_{\mathfrak{p}} X_1 \wedge \cdots \wedge X_k) = \sum_{i=1}^{k} (-1)^{i+1} (ZY_i \otimes_{\mathfrak{p}} X_1 \wedge \cdots \wedge \not{X_i} \wedge \cdots \wedge X_k)$$
$$+ \sum_{i<j} (-1)^{i+j} Z \otimes_{\mathfrak{p}} [[Y_i, Y_j]] \wedge X_1 \wedge \cdots \wedge \not{X_i} \wedge \cdots \wedge \not{X_j} \wedge \cdots \wedge X_k$$

Problem: Replacement for the commutator $[Y_i, Y_j]$ in the quantum case?

Observation: irreducible flag manifold, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p}$

$\Rightarrow$ hermitian symmetric space

$\Rightarrow$ $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{p}$

$\Rightarrow$ $[[Y_i, Y_j]] = 0$

$$\Rightarrow \delta_k(Z \otimes X_1 \wedge \cdots \wedge X_k) = \sum_{i=1}^{k} (-1)^{i+1} (ZY_i \otimes X_1 \wedge \cdots \wedge \not{X_i} \wedge \cdots \wedge X_k)$$

Quantum Bernstein–Gelfand–Gelfand Resolution

Quantum generalized Verma modules: $C_k := U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l})} \Lambda^k \mathfrak{m}$

Heckenberger/Kolb, 2007: There exists a complex

$$0 \longrightarrow C_n \xrightarrow{\delta_{n-1}} C_{n-1} \xrightarrow{\delta_{n-2}} \cdots \xrightarrow{\delta_1} C_1 \xrightarrow{\varepsilon} \mathbb{C} \longrightarrow 0, \quad n := \dim(\mathfrak{m})$$

given by unique embeddings of Verma modules

$$\delta_k(Z \otimes Y_{i_1} \wedge Y_{i_2} \wedge \cdots \wedge Y_{i_{k+1}}) = \sum_j \alpha_{i_j} Z Y_{i_j} \otimes Y_{i_1} \wedge \cdots \wedge \cancel{Y_{i_j}} \wedge \cdots \wedge Y_{i_{k+1}}$$

Dualizing: $C_k' := \left(U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l})} \Lambda^k \mathfrak{m}\right)' \supset \mathcal{O}(G_q) \,\square\, \Lambda^k \mathfrak{m}^*$

$\Rightarrow$ differential calculus: $\Omega^{(0,0)} \xrightarrow{\bar\partial_1} \Omega^{(0,1)} \xrightarrow{\bar\partial_2} \cdots \xrightarrow{\bar\partial_n} \Omega^{(0,n)}$

$$\Omega^{(0,k)} \cong \mathcal{O}(G_q) \,\square\, \Lambda^k \mathfrak{m}^* \subset C_k'$$

$$\bar\partial_k := \delta_k^* : C_{k-1}' \longrightarrow C_k'$$

$\Rightarrow$ Completely explicit description!

BGG resolution for quantum SO(5)/(SO(2)×SO(3))

$$U_q(\mathfrak{g}) \underset{U_q(\mathfrak{l})}{\otimes} \Lambda^3\mathfrak{m} \xrightarrow{\delta_3} U_q(\mathfrak{g}) \underset{U_q(\mathfrak{l})}{\otimes} \Lambda^2\mathfrak{m} \xrightarrow{\delta_2} U_q(\mathfrak{g}) \underset{U_q(\mathfrak{l})}{\otimes} \Lambda^1\mathfrak{m} \xrightarrow{\delta_1} U_q(\mathfrak{g}) \underset{U_q(\mathfrak{l})}{\otimes} \mathbb{C}$$

$$\delta_3(1 \otimes X_{-1} \wedge X_0 \wedge X_1) = X_{-1} \otimes X_0 \wedge X_1 - X_0 \otimes X_{-1} \wedge X_1 + \frac{1}{[3]_2} X_1 \otimes X_{-1} \wedge X_0$$

$$\delta_2(1 \otimes X_{-1} \wedge X_0) = [2]_1 X_0 \otimes X_0 - X_1 \otimes X_{-1} \qquad [n]_1 := \frac{q^n - q^{-n}}{q - q^{-1}}$$

$$\delta_2(1 \otimes X_{-1} \wedge X_1) = [2]_1 X_{-1} \otimes X_0 - X_0 \otimes X_{-1}, \qquad [n]_2 := \frac{q_2^n - q_2^{-n}}{q_2 - q_2^{-1}}$$

$$\delta_2(1 \otimes X_0 \wedge X_1) = [3]_2\, X_0 \otimes X_1 - X_1 \otimes X_0 \qquad q_2 := \sqrt{q}$$

$$\delta_1(Z_{-1} \otimes X_{-1} + Z_0 \otimes X_0 + Z_1 \otimes X_1) = Z_{-1}X_{-1} + Z_0 X_0 + Z_1 X_1$$

$$\Rightarrow \ \delta_{k-1} \circ \delta_k = 0.$$

Dualizing: $\Rightarrow \ \bar\partial_k := (\delta_k)^* : \Omega^{(0,k-1)} \longrightarrow \Omega^{(0,k)}$, $\bar\partial_{k+1} \circ \bar\partial_k = 0$

$$\begin{aligned}\Omega^{(0,k)} &= \mathcal{O}(SO_q(5)) \,\square\, \Lambda^k \mathfrak{m}^* \\ &= \Big\{ \sum f_i \otimes \omega_i : \sum (X \triangleright f_i) \otimes \omega_i = \sum f_i \otimes (\omega_i \triangleleft X),\ \forall X \in U_q(\mathfrak{l}) \Big\} \\ &\subset \quad \mathcal{O}(SO_q(5)) \otimes \Lambda^k \mathfrak{m}^*\end{aligned}$$

Matrix representation of the Dolbeault complex

$$\Omega^{(0,0)} = \mathcal{O}(SO_q(5)) \,\square\, \Lambda^0 \mathfrak{m}^* \subset \mathcal{O}(SO_q(5))$$

$$\Omega^{(0,1)} = \mathcal{O}(SO_q(5)) \,\square\, \Lambda^1 \mathfrak{m}^* \subset \mathcal{O}(SO_q(5)) \otimes \Lambda^1 \mathfrak{m}^* \cong \bigoplus_{i=1}^3 \mathcal{O}(SO_q(5))$$

$$\Omega^{(0,2)} = \mathcal{O}(SO_q(5)) \,\square\, \Lambda^2 \mathfrak{m}^* \subset \mathcal{O}(SO_q(5)) \otimes \Lambda^2 \mathfrak{m}^* \cong \bigoplus_{i=1}^3 \mathcal{O}(SO_q(5))$$

$$\Omega^{(0,3)} = \mathcal{O}(SO_q(5)) \,\square\, \Lambda^3 \mathfrak{m}^* \subset \mathcal{O}(SO_q(5))$$

$$\bar\partial_0 = \begin{pmatrix} X_{-1} \\ X_0 \\ X_1 \end{pmatrix}, \quad \bar\partial_1 = \begin{pmatrix} -X_0 & [2]_1 X_{-1} & 0 \\ -X_1 & [2]_1 X_0 & 0 \\ 0 & -X_1 & [3]_2 X_0 \end{pmatrix}, \quad \bar\partial_2 = \left(\tfrac{1}{[3]_2} X_1, -X_0, X_{-1}\right)$$

$\Rightarrow$ Bounded commutators!

Hermitian metric

Kähler geometry: Hodge $\star$-operator $\Leftrightarrow$ Hermitian inner product

$$\star\bar{\alpha} \wedge \beta = \langle\alpha,\beta\rangle_{\mathbb{C}}\,\mathrm{vol}, \text{ where } \Omega^{(n,n)} = B\cdot\mathrm{vol}$$

Observation: $\bar{\partial}_k$ unique up to a constant

$\Rightarrow$ May rescale $\langle\cdot,\cdot\rangle : \Omega^{(0,k)} \times \Omega^{(0,k)} \longrightarrow \mathbb{C}$.

Inner product: Set

$\langle\cdot,\cdot\rangle_h : \mathcal{O}(SO_q(5)) \times \mathcal{O}(SO_q(5)) \longrightarrow \mathbb{C}$, $\langle a,b\rangle_h := h(a^*b)$ (Haar state).

Choose $\langle\cdot,\cdot\rangle_k : \Lambda^k\mathfrak{m}^* \times \Lambda^k\mathfrak{m}^* \longrightarrow \mathbb{C}$ (orthogonal weight vectors).

Recall: $\Omega^{(0,k)} = \mathcal{O}(SO_q(5)) \,\square\, \Lambda^k\mathfrak{m}^* \subset \mathcal{O}(SO_q(5)) \otimes \Lambda^k\mathfrak{m}^*$

Define $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_h \otimes \langle\cdot,\cdot\rangle_k : \Omega^{(0,k)} \times \Omega^{(0,k)} \longrightarrow \mathbb{C}$.

Remark: $\bar{\partial}_k^\dagger \neq \bar{\partial}_k^*$, where $\langle\bar{\partial}_k(x),y\rangle = \langle x,\bar{\partial}_k^\dagger(y)\rangle$

Reason: $\bar{\partial}_k^*$ does not leave $(\Omega^{(0,k)})^\perp \subset \mathcal{O}(SO_q(5)) \otimes \Lambda^k\mathfrak{m}^*$ invariant!

$\Rightarrow$ need to apply orthogonal projection onto $\Omega^{(0,k)}$

Reducing computations to highest weight vectors

Unitary equivalence:

$$J_0 : {}^{\mu}_{\mu}V^{(0,0)}_{(0,0)} \longrightarrow {}^{\mu}_{\mu}\Omega^{(0,0)}, \quad J_0(a) := \tfrac{1}{c_0}\, a$$

$$J_1 : {}^{\mu}_{\mu}V^{(0,2)}_{(0,2)} \longrightarrow {}^{\mu}_{\mu}\Omega^{(0,1)}, \quad J_1(a) := \tfrac{1}{c_1}\Big(\tfrac{1}{[2]_2^2} F_2^2 \triangleright a \otimes \upsilon_{-1} + \tfrac{1}{[2]_2} F_2 \triangleright a \otimes \upsilon_0 + a \otimes \upsilon_1\Big)$$

$$J_2 : {}^{\mu}_{\mu}V^{(1,2)}_{(1,2)} \longrightarrow {}^{\mu}_{\mu}\Omega^{(0,2)}, \quad J_2(a) := \tfrac{1}{c_2}\Big(\tfrac{1}{[2]_2^2} F_2^2 \triangleright a \otimes \omega_{-1} + \tfrac{1}{[2]_2} F_2 \triangleright a \otimes \omega_0 + a \otimes \omega_1\Big)$$

$$J_3 : {}^{\mu}_{\mu}V^{(3,0)}_{(3,0)} \longrightarrow {}^{\mu}_{\mu}\Omega^{(0,3)}, \quad J_3(a) := \tfrac{1}{c_3}\, a \otimes \nu^0$$

$\Rightarrow$ $\eth_0 := J_1^* \circ \bar{\partial}_0 \circ J_0 \;:\; {}^{\mu}_{\mu}V^{(0,0)}_{(0,0)} \longrightarrow {}^{\mu}_{\mu}V^{(0,2)}_{(0,2)}, \quad \eth_0(v) = \frac{c_1}{c_0} X_1(v)$

$\eth_1 := J_2^* \circ \bar{\partial}_1 \circ J_1 \;:\; {}^{\mu}_{\mu}V^{(0,2)}_{(0,2)} \longrightarrow {}^{\mu}_{\mu}V^{(1,2)}_{(1,2)}, \quad \eth_1(v) = \frac{c_2}{c_1} \frac{[3]_2 - 1}{[3]_2} \mathrm{pr}_{(1,2)} \circ X_0(v)$

$\eth_2 := J_3^* \circ \bar{\partial}_2 \circ J_2 \;:\; {}^{\mu}_{\mu}V^{(1,2)}_{(1,2)} \longrightarrow {}^{\mu}_{\mu}V^{(3,0)}_{(3,0)}, \quad \eth_2(v) = \frac{c_3}{c_2} \mathrm{pr}_{(3,0)} \circ X_{-1}(v),$

with orthogonal projections

$$\mathrm{pr}_{(1,2)} := \big(1 - \tfrac{1}{[4]_2} F_2 E_2\big) \;:\; {}^{\mu}_{\mu}V^{(1,2)}_{(1,2)} \oplus {}^{\mu}_{\mu}V^{(0,4)}_{(1,2)} \longrightarrow {}^{\mu}_{\mu}V^{(1,2)}_{(1,2)}$$

$$\mathrm{pr}_{(n,0)} := \big(1 - \tfrac{1}{[2]_2} F_2 E_2 + \tfrac{1}{[2]_2^2 [3]_2} F_2^2 E_2^2\big) \;:\; {}^{\mu}_{\mu}V^{(n,0)}_{(n,0)} \oplus {}^{\mu}_{\mu}V^{(n-1,2)}_{(n,0)} \oplus {}^{\mu}_{\mu}V^{(n-2,4)}_{(n,0)} \longrightarrow {}^{\mu}_{\mu}V^{(n,0)}_{(n,0)}$$

Adjoint operators

$$\eth_0^\dagger : {}^\mu_\mu V^{(0,2)}_{(0,2)} \longrightarrow {}^\mu_\mu V^{(0,0)}_{(0,0)}, \quad \eth_0^\dagger(v) = \tfrac{c_1}{c_0}\,\mathrm{pr}_{(0,0)} \circ X_1^* \triangleright v$$

$$\eth_1^\dagger : {}^\mu_\mu V^{(1,2)}_{(1,2)} \longrightarrow {}^\mu_\mu V^{(0,2)}_{(0,2)}, \quad \eth_1^\dagger(v) = \tfrac{c_2}{c_1}\,\tfrac{[3]_2-1}{[3]_2}\,\mathrm{pr}_{(0,2)} \circ X_0^* \triangleright v$$

$$\eth_2^\dagger : {}^\mu_\mu V^{(3,0)}_{(3,0)} \longrightarrow {}^\mu_\mu V^{(1,2)}_{(1,2)}, \quad \eth_2^\dagger(v) = \tfrac{c_3}{c_2}\,X_{-1}^* \triangleright v$$

Observation: $u \in {}^\mu_\mu V^{\lambda_k}_{\lambda_k}$,

$\dim({}^\mu_\mu V^{\lambda_k}_{\lambda_k}) = 1 \;\Rightarrow\; \eth_k^\dagger \eth_k(u) = \rho_{\lambda_k} u, \quad \eth_{k-1}\eth_{k-1}^\dagger(u) = \sigma_{\lambda_{k-1}} u$

$\dim({}^\mu_\mu V^{\lambda_k}_{\lambda_k}) = 2 \;\Rightarrow\; \dim({}^\mu_\mu V^{\lambda_{k-1}}_{\lambda_{k-1}}) = \dim({}^\mu_\mu V^{\lambda_{k+1}}_{\lambda_{k+1}}) = 1$

$\mathrm{spec}(\eth_k^\dagger \eth_k)\backslash\{0\} = \mathrm{spec}(\eth_k \eth_k^\dagger))\backslash\{0\}$

$\Rightarrow$ It is enough to compute the spectrum of 1-dimensional operators!

$\Rightarrow$ We do not need to know explicitly the inner product!

The spectrum of the Dirac operator $D := \bar\partial + \bar\partial^\dagger$

$u \in {}^{\mu}_{\mu}V^{\lambda_k}_{\lambda_k}, \qquad \dim({}^{\mu}_{\mu}V^{\lambda_k}_{\lambda_k}) = 1, \qquad \eth_k^\dagger \eth_k(u) = \rho_{\lambda_k} u,$

$u_\pm := \sqrt{\rho_{\lambda_k}} J_k(u) \pm J_{k+1}(\eth_k(u)) \;\Rightarrow\; D(u_\pm) = \pm\sqrt{\rho_{\lambda_k}}\, u_\pm$

$v \in {}^{\mu}_{\mu}V^{\lambda_{k+1}}_{\lambda_{k+1}}, \quad \dim({}^{\mu}_{\mu}V^{\lambda_{k+1}}_{\lambda_{k+1}}) = 1, \quad \eth_k \eth_k^\dagger(v) = \sigma_{\lambda_k} v,$

$v_\pm := \sqrt{\sigma_{\lambda_k}} J_k(v) \pm J_{k+1}(\eth_k^\dagger(v)) \;\Rightarrow\; D(v_\pm) = \pm\sqrt{\sigma_{\lambda_k}}\, v_\pm$

k=0: $\rho_{(2n,2l)} := \frac{c_1^2}{c_0^2}\,\frac{q^2\,[2]_2^2\,([n+l+2]_1[n+l]_1+[l+1]_1[l]_1)}{[3]_2},$

k=1: $\rho_{(2n+1,2l)} := \frac{c_2^2}{c_1^2}\,\frac{q^2\,[2]_2\,([3]_2-1)^2\,(\,[n+l+2]_1[n+l+1]_1+[n+2]_1[n]_1)}{[3]_2^2\,[2]_1^2},$

k=2: $\sigma_{(2n,2l)} := \frac{c_2^2}{c_1^2}\,\frac{q^2\,[2]_2\,([3]_2-1)^2\,(\,[n+l+2]_1[n+l]_1+[n+1]_1[n]_1)}{[3]_2^2\,[2]_1^2},$

k=3: $\sigma_{(2n+1,2l)} := \frac{c_3^2}{c_2^2}\,\frac{q^2\,([n+l+2]_1[n+l+1]_1+[n+2]_1[n]_1)}{[3]_2}.$

Multiplicities: $\dim\left({}^{(n,l)}V^{\lambda_k}_{\lambda_k}\right) = \frac{1}{6}(n+1)(l+1)(n+l+2)(2n+l+3)$

$[n]_1 := \frac{q^n-q^{-n}}{q-q^{-1}}, \qquad [n]_2 := \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}$

Branching laws

Peter-Weyl Theorem: $\mathcal{O}(SO_q(5)) = \bigoplus_{(n,l)\in\mathbb{N}_0\times 2\mathbb{N}_0} V^{(n,l)*} \otimes V^{(n,l)}$

$\sum f_i \otimes \omega_i \in \Omega^{(0,k)} \Leftrightarrow \sum (X \triangleright f_i) \otimes \omega_i = \sum f_i \otimes (\omega_i \triangleleft X) \quad \forall X \in U_q(\mathfrak{l})$

$\Rightarrow$ $\mathrm{span}\{f_i : i = 1, ..., \dim(\Lambda^k \mathfrak{m}^*)\}$ and $\mathrm{span}\{\omega_i : i = 1, ..., \dim(\Lambda^k \mathfrak{m}^*)\}$ are the same (dual) representations of $U_q(\mathfrak{l})$

Branching laws: $V(m,k)$:= irred. rep. of $U_q(\mathfrak{l})$ of highest weight (m,k)

- $\mathbf{\Omega^{(0,0)}}$: $V(0,0) \subset V^{(n,l)}$
- $\mathbf{\Omega^{(0,1)}}$: $V(0,2) \subset V^{(n,l)}$
- $\mathbf{\Omega^{(0,2)}}$: $V(1,2) \subset V^{(n,l)}$
- $\mathbf{\Omega^{(0,3)}}$: $V(3,0) \subset V^{(n,l)}$

Notation: ${}^{\mu}V^{\lambda}$:= irred. right $U_q(\mathfrak{g})$ and left $U_q(\mathfrak{l})$ of h. w. μ and λ resp.

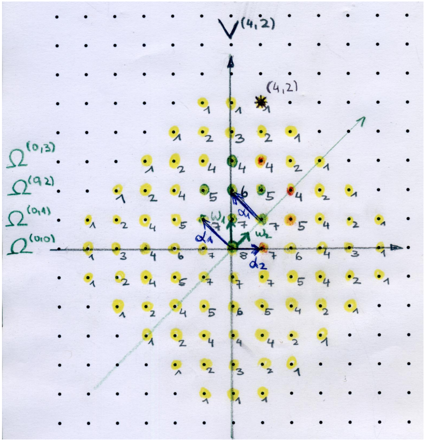
${}^{\mu}_{\nu}V^{\lambda}_{\rho}$:= corresponding weight spaces

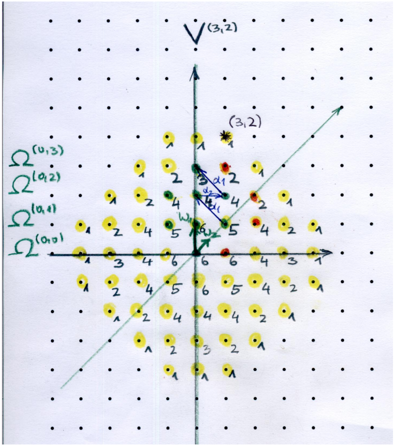
Branching laws

Peter-Weyl Theorem: $\mathcal{O}(SO_q(5)) \cong \bigoplus_{(n,l)\in\mathbb{N}_0\times 2\mathbb{N}_0} V^{(n,l)*} \otimes V^{(n,l)}$

Branching law: How many times occurs $V(\lambda_k) \cong \Lambda^k \mathfrak{m}^*$ in $V^{(n,l)}$?

Answer: $\dim\big(V^{(n,l)}_{\lambda_k}\big) - \dim\big(V^{(n,l)}_{\lambda_k+\alpha_2}\big)$, $\lambda_k = (0,0), (0,2), (1,2), (3,0)$





Branching law: result for $U_q(\mathfrak{l}) \subset U_q(\mathrm{so}(5))$

$\Omega^{(0,0)}$: $V(0,0) \subset V^{(n,l)}$

- multiplicity 1 in $V^{(2n,2l)}$

$\Omega^{(0,1)}$: $V(0,2) \subset V^{(n,l)}$

- multiplicity 2 in $V^{(2n+2,2l+2)}$,
- multiplicity 1 in $V^{(2n+1,2l+2)}$, $V^{(2n+2,0)}$ and $V^{(0,2l+2)}$

$\Omega^{(0,2)}$: $V(1,2) \subset V^{(n,l)}$

- multiplicity 2 in $V^{(2n+3,2l+2)}$,
- multiplicity 1 in $V^{(2n+2,2l+2)}$, $V^{(2n+3,0)}$ and $V^{(1,2l+2)}$

$\Omega^{(0,3)}$: $V(3,0) \subset V^{(n,l)}$

- multiplicity 1 in $V^{(2n+3,2l)}$,

- multiplicity 0 in all other cases, where $n, l \in \mathbb{N}_0$.

Cohomology: $0 \longrightarrow V^{(2n,2l)}_{(0,0)} \xrightarrow{\eth_0} V^{(2n,2l)}_{(0,2)} \xrightarrow{\eth_1} V^{(2n,2l)}_{(1,2)} \longrightarrow 0$ and

$0 \longrightarrow V^{(2n+1,2l)}_{(0,0)} \xrightarrow{\eth_1} V^{(2n+1,2l)}_{(0,2)} \xrightarrow{\eth_2} V^{(2n+1,2l)}_{(1,2)} \longrightarrow 0$ exact but for $V^{(0,0)}_{(0,0)}$

Highest weight vectors

1. $${}^{(2n,2l)}_{(2n,2l)}V^{(0,0)}_{(0,0)} = \mathrm{span}\{(u^1_1u^1_5)^n(u^1_1u^2_5 - qu^2_1u^1_5)^l\}, \;\; n,l \in \mathbb{N}_0.$$

2. $${}^{(2n,2l)}_{(2n,2l)}V^{(0,2)}_{(0,2)} = \mathrm{span}\Big\{(u^1_1u^1_5)^{n-1}u^1_4u^1_5(u^1_1u^2_5 - qu^2_1u^1_5)^l, \\ (u^1_1u^1_5)^n(u^1_4u^2_5 - qu^2_4u^1_5)(u^1_1u^2_5 - qu^2_1u^1_5)^{l-1}\Big\}, \;\; n>0, \;\; l>0,$$
$${}^{(2n,0)}_{(2n,0)}V^{(0,2)}_{(0,2)} = \mathrm{span}\{(u^1_1u^1_5)^{n-1}u^1_4u^1_5\}, \;\; n>0,$$
$${}^{(0,2l)}_{(0,2l)}V^{(0,2)}_{(0,2)} = \mathrm{span}\{(u^1_4u^2_5 - qu^2_4u^1_5)(u^1_1u^2_5 - qu^2_1u^1_5)^{l-1}\}, \;\; l>0,$$
$${}^{(2n+1,2l)}_{(2n+1,2l)}V^{(0,2)}_{(0,2)} = \mathrm{span}\Big\{(u^1_1u^1_5)^{n-1}\Big(u^1_3(u^1_4u^2_5 - qu^2_4u^1_5) \\ -qu^1_4(u^1_3u^2_5 - qu^2_3u^1_5)\Big)(u^1_1u^2_5 - qu^2_1u^1_5)^{l-1}\Big\}, \;\; n \in \mathbb{N}_0, \;\; l>0.$$

3. $${}^{(2n,2l)}_{(2n,2l)}V^{(1,2)}_{(1,2)} = \mathrm{span}\Big\{(u^1_1u^1_5)^{n-1}\Big(u^1_3u^1_5(u^1_4u^2_5 - qu^2_4u^1_5) \\ -qu^1_4u^1_5(u^1_3u^2_5 - qu^2_3u^1_5)\Big)(u^1_1u^2_5 - qu^2_1u^1_5)^{l-1}\Big\}, \;\; n>0, \;\; l>0,$$
$${}^{(2n+1,2l)}_{(2n+1,2l)}V^{(1,2)}_{(1,2)} = \mathrm{span}\Big\{(u^1_1u^1_5)^{n-1}u^1_4(u^1_5)^2(u^1_1u^2_5 - qu^2_1u^1_5)^l, \\ (u^1_1u^1_5)^n u^1_5(u^1_4u^2_5 - qu^2_4u^1_5)(u^1_1u^2_5 - qu^2_1u^1_5)^{l-1}\Big\}, \;\; n>0, \;\; l>0,$$
$${}^{(2n+1,0)}_{(2n+1,0)}V^{(1,2)}_{(1,2)} = \mathrm{span}\{(u^1_1u^1_5)^{n-1}u^1_4(u^1_5)^2\}, \;\; n>0,$$
$${}^{(1,2l)}_{(1,2l)}V^{(1,2)}_{(1,2)} = \mathrm{span}\{u^1_5(u^1_4u^2_5 - qu^2_4u^1_5)(u^1_1u^2_5 - qu^2_1u^1_5)^{l-1}\}, \;\; l>0.$$

4. $${}^{(2n+1,2l)}_{(2n+1,2l)}V^{(3,0)}_{(3,0)} = \mathrm{span}\{(u^1_1u^1_5)^{n-1}(u^1_5)^3(u^1_1u^2_5 - qu^2_1u^1_5)^l\}, \;\; n>0, \;\; l \in \mathbb{N}_0.$$