Spectral gaps for higher Laplacians and group cohomology

Piotr Mizerka

Institute of Mathematics of Polish Academy of Sciences

Noncommutative geometry: metric and spectral aspects,
Kraków, 28 September 2022

Motivation

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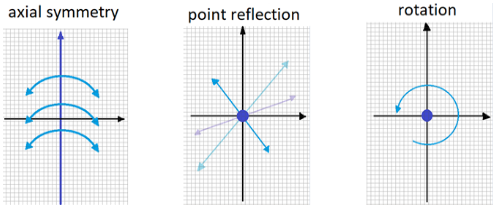

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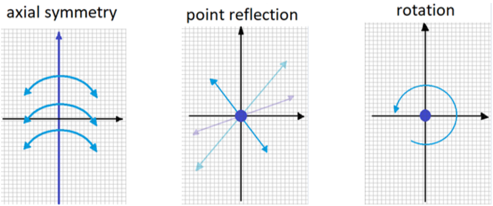

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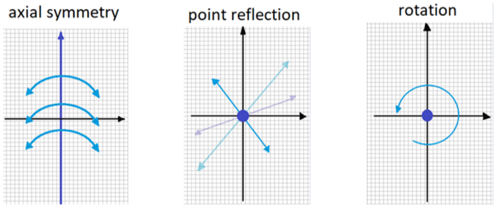

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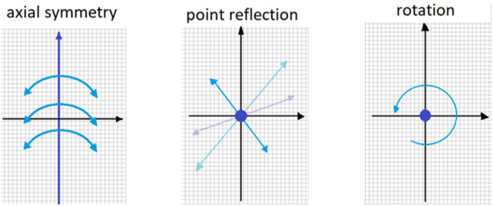

- Generalization: symmetries must have fixed points
- An "isometric" fixed point property: Kazhdan's Property (T)
- (T) can be applied to construct expanders

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- Idea: interpretation in a group ring setting

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- Spectral gap for the first Laplacian of $SL_3(\mathbb{Z})$

Vanishing and reducibility of cohomology and property (T)

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- there are several ways to compute group cohomology
- one may use e.g. projective resolutions:

$$\mathcal{F} = \cdots F_n \to \cdots \to F_0 \to \mathbb{Z} \to 0,$$

$$H^n(G, V) = H_n(\mathrm{Hom}_G(\mathcal{F}, V))$$

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Theorem (Ozawa, 2014)

$G = \langle s_1, \ldots, s_n | \cdots \rangle$ has property (T) iff there exists $\lambda > 0$ such that $\Delta_0^2 - \lambda\Delta_0 = \sum \xi_i^* \xi_i$ ($\Delta_0 = d_0^* d_0 = \sum_{i=1}^n (1 - s_i)^*(1 - s_i)$).

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Definition

The ith reduced cohomology is defined by $\overline{H}^i = \operatorname{Ker} d_i / \overline{\operatorname{Im} d_{i-1}}$.
We say that the ith cohomology is reduced if H^i coincides with $\overline{H}^i$.

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Proposition (Dymara-Januszkiewicz)

For any $i \geq 2$ there exists a group G_i with reduced H^i and $H^i(G, \rho_0) \neq 0$ for some unitary representation ρ_0.

Spectral gaps for higher Laplacians vs vanishing and reducibility

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Definition

$M \in M_n(\mathbb{R}G)$ is an SOS if there exist $M_1, \ldots, M_l$ such that

$$M = M_1^* M_1 + \cdots + M_l^* M_l.$$

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$$\cdots \to (\mathbb{Z}G)^{k_{i-1}} \xrightarrow{d_{i-1}} (\mathbb{Z}G)^{k_i} \xrightarrow{d_i} (\mathbb{Z}G)^{k_{i+1}} \to \cdots$$

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TFAE for G and $i \geq 1$:

- *H^i vanish and H^{i+1} are reduced.*
- *$\Delta_i - \lambda I = \mathrm{SOS}$ for some $\lambda > 0$.*

How to get the matrices d_i?

Fox calculus

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The differentials $\frac{\partial}{\partial s_j} : \mathbb{R}F_n \to \mathbb{R}G$, $F_n = \langle s_1, \ldots, s_n \rangle$ are defined by:

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The Fox derivatives are the elements $\frac{\partial r_i}{\partial s_j} \in \mathbb{R}G$.

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Theorem (Lyndon, '50s)

The cohomology $H^*(G, V)$ is the cohomology of the following complex:

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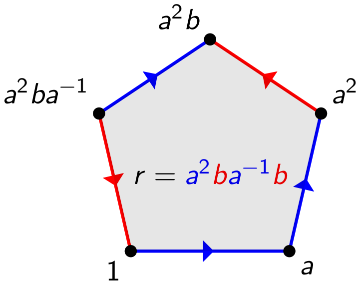

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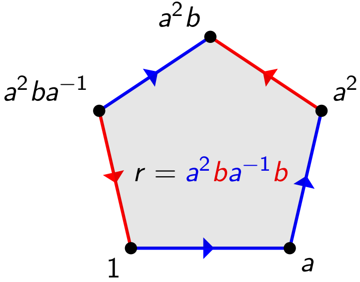

- $0 \to V \xrightarrow{\begin{bmatrix} 1-a \\ 1-b \end{bmatrix}} V^2 \xrightarrow{\begin{bmatrix} 1 + a - a^2 b a^{-1} & a^2 + a^2 b a^{-1} \end{bmatrix} = d_1} V$

Spectral gap for the first Laplacian of $SL_3(\mathbb{Z})$

(joint work with M. Kaluba and P. Nowak)

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Lemma

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- Convex optimization for $M = \Delta_1 - \lambda I$:

$$\begin{aligned} \text{maximize:} \quad & \lambda \\ \text{subject to:} \quad & M_{i,j}(g) = \langle \delta_{i,j} \otimes \delta_g, P \rangle, \\ & P \succeq 0. \end{aligned}$$

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Theorem (Kaluba, M., Nowak)

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Corollary

The first cohomology of $SL_3(\mathbb{Z})$ vanishes, and the second is reduced.

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- Expanders generalize to higher dimensions (Lubotzky)
- $SL_3(\mathbb{Z})$: spectral gap $\Rightarrow$ "CW-expanders"

Thank you for attention!