Quantum Kaluza-Klein Theory with $M_2(C)$ and particle physics

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Noncommutative geometry: metric and spectral aspects

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Elements of Quantum Riemannian Geometry(QRG)

Quantum Riemannian Geometry on $M_2(C)$

Kaluza-Klein model and scalar fields on $C^\infty(M)\otimes M_2(C)$

Elements of Quantum Riemannian Geometry(QRG)

We work with an algebra A, typically a $*$-algebra over C.

Differential structure d: $A \to \Omega^1$ $\quad$ $d(ab) = adb + (da)b$, where $\Omega^1 = \text{span}\{a\, db \mid a, b \in A\}$

$d: \Omega^n \to \Omega^{n+1}$ $\quad$ $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^n\, \omega \wedge d\eta$, $d^2 = 0$

where $\omega, \eta \in \Omega = \bigoplus_{n \geq 0} \Omega^n$ with $\Omega^0 = A$

Connection $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$, $\quad$ $\nabla(a.\omega) = a.\nabla\omega + da \otimes_A \omega$, $\quad$ where $\omega \in \Omega^1$

σ map: $\Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$, $\quad$ $\nabla(\omega.a) = (\nabla\omega).a + \sigma(\omega \otimes_A da)$

Connection $\nabla: \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1 \otimes_A \Omega^1$ $\quad$ $\nabla(\omega \otimes_A \eta) = \nabla\omega \otimes_A \eta + (\sigma \otimes_A \text{id})(\omega \otimes_A \nabla\eta)$

Elements of Quantum Riemannian Geometry(QRG)

Metric $g \in \Omega^1 \otimes_A \Omega^1$, which is invertible in the sense of a bimodule map $(\ ,\ ): \Omega^1 \otimes_A \Omega^1 \to A$ obeying

$$((\omega, \ ) \otimes_A \mathrm{id})g = \omega = (\mathrm{id} \otimes_A (\ , \omega))g$$

quantum symmetry condition: $\wedge(g)=0$

Quantum Levi-Civita connection(QLC): $\nabla g = 0$ and $\wedge\nabla - d=0$

Riemann curvature $R_\nabla : \Omega^1 \to \Omega^2 \otimes_A \Omega^1$ $R_\nabla = (d \otimes_A \mathrm{id} - \mathrm{id} \wedge \nabla)\nabla$

Ricci tensor : $\mathrm{Ricci} = ((\ ,\ ) \otimes_A \mathrm{id})(\mathrm{id} \otimes i \otimes \mathrm{id})(\mathrm{id} \otimes R_\nabla)g$, where $i: \Omega^2 \to \Omega^1 \otimes_A \Omega^1$

Ricci scalar S: $S = (\ ,\ )\mathrm{Ricci} \in A$

Reality condition: $g^\dagger = g$ and $\nabla \circ * = \sigma \circ \dagger \circ \nabla$

Recall QRG on algebra A = $M_2(C)$

In terms of a self-adjoint basis s^i (where $s^{i*} = s^i$), the exterior algebra is

$$s^1 \wedge s^1 = s^2 \wedge s^2 = \imath \mathrm{Vol}, \quad s^1 \wedge s^2 = s^2 \wedge s^1 = 0$$

Using Pauli matrices and $1 = \mathrm{id}$ as a basis of $M_2(\mathbb{C})$, the differentials are then

$$\mathrm{d}\sigma^1 = \sigma^3 s^2, \quad \mathrm{d}\sigma^2 = -\sigma^3 s^1, \qquad \mathrm{d}\sigma^3 = \sigma^2 s^1 - \sigma^1 s^2, \quad \mathrm{d}s^i = -\sigma^i \mathrm{Vol}, \quad \mathrm{Vol} = -\imath s^1 \wedge s^1$$

The quantum metric of interest is

$$g = -s^1 \otimes s^1 + s^2 \otimes s^2$$

A natural 1-parameter QLC has braiding (where $\bar{1} = 2$ and $\bar{2} = 1$),

$$\sigma(s^i \otimes s^j) = \begin{cases} s^i \otimes s^j & \text{if } i \neq j \\ -s^{\bar{i}} \otimes s^{\bar{i}} - 2\imath\rho s^{\bar{i}} \otimes s^i & \text{if } i = j \end{cases}$$

Recall QRG on algebra A = $M_2(C)$

The connection is then

$$\nabla s^{i} = \frac{\imath}{2}\sigma^{i}(s^{1}\otimes s^{1}+s^{2}\otimes s^{2}) + \frac{\imath}{2}\epsilon_{ij}\sigma^{j}(s^{2}\otimes s^{1}-s^{1}\otimes s^{2}) - \rho\sigma^{i}s^{\bar{i}}\otimes s^{i}.$$

Choosing the canonical symmetric lift

$$i(\mathrm{Vol}) = \frac{1}{2\imath}(s^{1}\otimes s^{1}+s^{2}\otimes s^{2})$$

Riemann curvature and Ricci tensor are

$$R_{\nabla}s^{i} = -\imath(1+\rho^{2})\mathrm{Vol}\otimes s^{i}, \quad \mathrm{Ricci} = -\frac{1}{2}(1+\rho^{2})(s^{1}\otimes s^{1}+s^{2}\otimes s^{2}), \quad S = 0.$$

QRG on algebra A = $\boldsymbol{C}^{\infty}(\boldsymbol{M}) \otimes \boldsymbol{M}_2(\boldsymbol{C})$

Follow a quantum Kaluza-Klein formulation, we solve for the quantum Riemannian geometry on A = $\boldsymbol{C}^{\infty}(\boldsymbol{M}) \otimes \boldsymbol{M}_2(\boldsymbol{C})$,

We take the graded tensor product exterior algebra $\Omega(A) = \Omega(M) \underline{\otimes} \Omega(M_2(\mathbb{C}))$

Proposition: For the reason that quantum metric g has to be central, the most general metric has the form

$$g = g_{\mu\nu}(x,t)\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} + A_{i\mu}(x,t)(s^i \otimes \mathrm{d}x^{\mu} + \mathrm{d}x^{\mu} \otimes s^i) + h_{ij}(x,t)s^i \otimes s^j$$

where $g_{\mu\nu}$ are symmetric, $h_{11} + h_{22} = 0$ and $A_{i\mu} = A_{\mu i}$, and all coefficients are functions on spacetime M.

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QRG on algebra A = $C^{\infty}(M) \otimes M_2(C)$

From now on, we assume for simplicity that if σ exists, it obeys

(i) $\sigma(\mathrm{d}x^\mu \otimes s^i)$ is the flip;

(ii) The result of $\sigma(s^i \otimes s^j)$ doesn't have any terms with $\mathrm{d}x^\mu$.

Proposition: If ∇ is a torsion free bimodule connection then it has the form

$$\nabla \mathrm{d}x^\mu = -\Gamma^\mu_{\alpha\beta}\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta + B^\mu_{i\alpha}(\mathrm{d}x^\alpha \otimes s^i + s^i \otimes \mathrm{d}x^\alpha) + C^\mu_{ij}s^i \otimes s^j$$
$$\nabla s^k = D^k_{\alpha\beta}\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta + E^k_{i\alpha}(\mathrm{d}x^\alpha \otimes s^i + s^i \otimes \mathrm{d}x^\alpha) + \gamma^k_{ij}s^i \otimes s^j$$

where $\Gamma^\mu_{\alpha\beta}, D^k_{\alpha\beta}$ are symmetric for the subscripts, $\Gamma^\mu_{\alpha\beta}, B^\mu_{i\alpha}, C^\mu_{ij}, D^k_{\alpha\beta}, E^k_{i\alpha} \in C^\infty(M)$, $\gamma^k_{ij} \in C^\infty(M) \otimes M_2(\mathbb{C})$ but does not have an σ^3 component, and

$$C^\mu_{11} + C^\mu_{22} = 0, \quad \gamma^k_{11} + \gamma^k_{22} = \imath\sigma^k.$$

QRG on algebra A = $C^\infty(M)\otimes M_2(C)$

Proposition: The only metrics that admit a QLC are of the form

$$g = g_{\mu\nu}(x,t)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + h_{ij}(x,t)s^i \otimes s^j$$

and the QLC has the form in last proposition with the further restrictions $D = B = 0$, Γ is the usual LC for $g_{\mu\nu}$, γ is the usual QLC for h_{ij} in the sense

$$h_{ji}\gamma^j_{mn} + 2\imath h_{pj}\gamma^p_{mnq}\gamma^j_{qi} + h_{nj}\gamma^j_{mi} = 0$$

and C, E obey

$$C^\mu_{ij} = -g^{\mu\alpha}h_{mj}E^m_{i\alpha},$$
$$\partial_\alpha h_{ij} + h_{mj}E^m_{i\alpha} + h_{im}E^m_{j\alpha} = 0,$$
$$2\imath h_{pm}E^m_{n\alpha}\gamma^p_{ijn} + (h_{jm} - h_{mj})E^m_{i\alpha} = 0.$$

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The inverse of the metric (,) has the form

$$(\mathrm{d}x^\mu, \mathrm{d}x^\nu) = g^{\mu\nu}, \quad (s^i, s^j) = h^{ij}, \quad (\mathrm{d}x^\mu, s^i) = 0, \quad (s^i, \mathrm{d}x^\mu) = 0$$

where $g^{\mu\nu}, h^{ij}$ are the inverse of $g_{\mu\nu}, h_{ij}$

The Laplacian on $f = f_a(x,t)\sigma^a$ is

$$\Delta f = (\ ,\ )\nabla \mathrm{d} f = \Delta_{LB} f + \Delta_{M_2} f - g^{\mu\alpha} h_{mj} E^m_{i\alpha} (\partial_\mu f_a)\sigma^a h^{ij}$$

where Δ_{LB} is the usual GR Laplacian on each component f_a and

$$\Delta_{M_2} f = h^{ij}(f_i \sigma^j - \epsilon_{kcb} f_c \sigma^b \gamma^k_{ij})$$

is the Laplacian on $M_2(\mathbb{C})$ at each x, t.

QRG on algebra A = $C^\infty(M) \otimes M_2(C)$

If $h_{ij} s^i \otimes s^j = h(x,t)(-s^1 \otimes s^1 + s^2 \otimes s^2)$, then

$$\Delta f(x,t) = (\Delta_{LB} f_a(x,t))\sigma^a + \frac{1}{h(x,t)}\left(-f_1(x,t)\sigma^1 + f_2(x,t)\sigma^2 - g^{\mu\nu}(\partial_\nu h(x,t)\partial_\mu f_a(x,t))\sigma^a\right).$$

Especially, if $h(x,t)$ is constant in spacetime and we let $h^{-1}(x,t) = \delta_m$ then the KG equation operator reduces to

$$(\Delta + m^2)f = (\Delta_{LB} f_a)\sigma^a + m^2 f_0 + (m^2 - \delta_m) f_1(x,t)\sigma^1 + (m^2 + \delta_m) f_2(x,t)\sigma^2 + m^2 f_3 \sigma^3,$$

This implies the field f usually regarded as one component field in classical spacetime now can be regarded as quadruplet of fields having different masses.

QRG on algebra A = $C^{\infty}(M) \otimes M_2(C)$

Proposition:

If the $h_{ij}s^i \otimes s^j = h(x,t)(-s^1 \otimes s^1 + s^2 \otimes s^2)$ and we let $h(x,t) = e^{\frac{2}{\sqrt{3-3\rho^2}}\phi(x,t)}$ then

$$S = -\frac{1}{2}S_M + (\nabla^\mu \phi)(\nabla_\mu \phi) + \frac{2}{\sqrt{3-3\rho^2}}\nabla^\mu \nabla_\mu \phi$$

where $\phi(x,t)$ is real scalar field and ρ is an imaginary parameter.

From here we can see that this result has the scalar field which also appears in KK theory.

Thank you

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