Atiyah sequences of braided Lie algebras and their splittings

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Noncommutative geometry: metric and spectral aspects:

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recent papers

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Abstract

Try to work out a gauge algebroid for a noncommutative principal bundle

Try to get a suitable class of (infinitesimal) gauge transformations

some natural structures

braiding Lie algebras to get bigger classes

a sequenze of braided Lie algebras; its splitting as a connection

Weil algebra

Chern–Weil homomorphism and braided Lie algebra cohomology

upgrade it to Hopf algebra cyclic cohomology

The classical gauge groupoid   C. Ehresmann, J. Pradines

$\pi : P \to M$ a G-principal bundle over M

the diagonal action of G on $P \times P$ given by $(u,v)g := (ug, vg)$

$[u,v]$ the orbit of (u,v) and $\Omega = P \times_G P$ the collection of orbits

Ω is a groupoid over M, — the *gauge* or *Ehresmann groupoid*

- source and target projections:

$$s([u,v]) := \pi(v), \qquad t([u,v]) := \pi(u).$$

- object inclusion: $M \to P \times_G P$, $m \mapsto \mathrm{id}_m := [u,u]$, u any element in $\pi^{-1}(m)$.

- partial multiplication: $[u,v'] \cdot [v,w]$, defined when $\pi(v') = \pi(v)$,

$$[u,v] \cdot [v',w] = [u, wg],$$

for the unique $g \in G$ such that $v = v'g$.

- the inverse:

$$[u,v]^{-1} = [v,u].$$

A *bisection* of the groupoid Ω is a map $\sigma : M \to \Omega$,

which is right-inverse to the source $s \circ \sigma = \mathrm{id}_M$

and such that $t \circ \sigma : M \to M$ is a diffeomorphism.

The collection $\mathcal{B}(\Omega)$ of bisections, form a group:

- multiplication: $\sigma_1 * \sigma_2(m) := \sigma_1\big((t \circ \sigma_2)(m)\big)\sigma_2(m)$
- identity id: the object inclusion $m \mapsto \mathrm{id}_m$.
- inverse: $\sigma^{-1}(m) = \big(\sigma\big((t \circ \sigma)^{-1}(m)\big)\big)^{-1}$

here $(t \circ \sigma)^{-1}$ as a diffeo of M, while the second inversion is the one in Ω.

The subset $\mathcal{B}_{P/G}(\Omega)$ of 'vertical' bisections, those that are right-inverse to the target projection as well, $t \circ \sigma = \mathrm{id}_M$, form a subgroup of $\mathcal{B}(\Omega)$.

A classical result:

• a group isomorphism between $\mathcal{B}(\Omega)$ and the group of principal (G-equivariant) bundle automorphisms of the principal bundle,

$$\mathrm{Aut}_G(P) := \{\varphi : P \to P \,;\; \varphi(pg) = \varphi(p)g\}\,,$$

• while $\mathcal{B}_{P/G}(\Omega)$ is isomorphic to the subgroup of gauge transformations, bundle vertical automorphisms (project to the identity on the base space):

$$\mathrm{Aut}_{P/G}(P) := \{\varphi : P \to P \,;\; \varphi(pg) = \varphi(p)g\,,\; \pi(\varphi(p)) = \pi(p)\}.$$

at level of groups

$$1 \to \mathsf{Aut}_{P/G}(P) \to \mathsf{Aut}_G(P) \to \mathsf{Diff}(M) \to 1$$

at level of derivations

$$0 \to \mathcal{X}(P)^{ver}_G \to \mathcal{X}(P)_G \to \mathcal{X}(M) \to 0$$

a splitting of this sequence is a way to give a connection

(horizontal lift or a vertical projection)

Noncommutative principal bundles

- H a Hopf algebra
- A a right H-comodule algebra with coaction $\delta^A : A \to A \otimes H$; $\quad \delta(a) = a_{(0)} \otimes a_{(1)}$

$\Rightarrow$ the subalgebra of coinvariant elements

$$B := A^{coH} = \left\{ b \in A \mid \delta^A(b) = b \otimes 1_H \right\}$$

The extension $B \subseteq A$ is H-Hopf–Galois if the canonical Galois map

$$\chi : A \otimes_B A \longrightarrow A \otimes H, \quad a' \otimes_B a \mapsto a' a_{(0)} \otimes a_{(1)}$$

is an isomorphism

χ is left A-linear, its inverse is determined by the restriction $\tau := \chi^{-1}_{|_{1_A \otimes H}}$

$$\tau = \chi^{-1}_{|_{1_A \otimes H}} : H \to A \otimes_B A \ , \quad h \mapsto \tau(h) = h^{<1>} \otimes_B h^{<2>} \, .$$

the translation map; thus by definition:

$$h^{<1>} h^{<2>}{}_{(0)} \otimes h^{<2>}{}_{(1)} = 1_A \otimes h$$

Everything algebraic

G be a semisimple affine algebraic group

$\pi : P \to P/G$ be a principal G-bundle with P and P/G affine varieties

$H = \mathcal{O}(G)$ the dual coordinate Hopf algebra

$A = \mathcal{O}(P)$, $B = \mathcal{O}(P/G)$ the dual coordinate algebras

$B \subseteq A$ be the subalgebra of functions constant on the fibers.

Then $B = A^{coH}$ and $\mathcal{O}(P \times_{P/G} P) \simeq A \otimes_B A$

Bijectivity of $P \times G \to P \times_{P/G} P$, $(p,g) \mapsto (p,pg)$, characterizing principal bundles, corresponds to the bijectivity of the canonical map $\chi : A \otimes_B A \to A \otimes H$

thus $B = A^{coH} \subseteq A$ is a Hopf–Galois extension

An important notion is that of the classical translation map

$t : P \times_{P/G} P \to G, \quad (p,q) \mapsto t(p,q)$ where $q = p\, t(p,q)$

the dual to τ before

Gauge transformations

T. Brzeziński: gauge transformations as invertible and unital comodule maps, with no additional requirement, i.e. not asked to be algebra maps;

The resulting gauge group might be very big; for example the gauge group of a G-bundle over a point would be much bigger than the structure group G

P. Aschieri, GL, CPagani: gauge transformations are taken to be algebra homomorphisms; this property implies in particular that they are invertible;

The resulting gauge group might be in general very small;

It works in a quasi-commutative context for the algebra A, with H a co-quasitriangular Hopf algebra; the base space algebra B in the centre of A

Gauge symmetry as (infinite dimensional) braided Hopf algebra of symmetry

The classical (commutative) case

The group $\mathcal{G}_P$ of gauge transformations of a principal G-bundle $\pi : P \to P/G$ is the group (for point-wise product) of G-equivariant maps

$$\mathcal{G}_P := \{\sigma : P \to G;\ \sigma(pg) = g^{-1}\sigma(p)g\}$$

Equivalently, is the subgroup (for map composition) of principal bundle automorphisms which are vertical (project to the identity on the base space):

$$\mathrm{Aut}_{P/G}(P) := \{\varphi : P \to P;\ \varphi(pg) = \varphi(p)g,\ \pi(\varphi(p)) = \pi(p)\},$$

These definitions can be dualised for algebras rather than spaces.

For $A = \mathcal{O}(P)$, $B = \mathcal{O}(P/G)$, $H = \mathcal{O}(G)$, the gauge group $\mathcal{G}_P$ of G-equivariant maps corresponds to H-equivariant maps that are also algebra maps

$$\mathcal{G}_A := \{\mathsf{f} : H \to A;\ \delta^A \circ \mathsf{f} = (\mathsf{f} \otimes \mathsf{id}) \circ \mathsf{Ad}\,,\ \mathsf{f}\ \text{algebra map}\}\ .$$

The group structure is the convolution product.

Similarly, the vertical automorphisms description leads to H-equivariant maps

$$\mathsf{Aut}_B\, A = \{\mathsf{F} : A \to A;\ \delta^A \circ \mathsf{F} = (\mathsf{F} \otimes \mathsf{id}) \circ \delta^A\,,\ \mathsf{F}|_B = \mathsf{id} : B \to B\,,\ \mathsf{F}\ \text{algebra map}\}\ .$$

The noncommutative case

Let $B = A^{coH} \subseteq A$ be a faithfully flat Hopf–Galois extension

The collection $\mathrm{Aut}_H(A)$ of unital algebra maps of A into itself, which are H-equivariant,

$$\delta^A \circ \mathsf{F} = (\mathsf{F} \otimes \mathrm{id}) \circ \delta^A \qquad F(a)_{(0)} \otimes F(a)_{(1)} = F(a_{(0)}) \otimes a_{(1)}$$

and restrict to the identity on the subalgebra B, is a group by map composition

with inverse operation

$$F^{-1}(a) = a_{(0)} F({a_{(1)}}^{<1>})\, {a_{(1)}}^{<2>}$$

H.P. Schneider: vertical H-equivariant algebra maps are invertible

Bialgebroids

B an algebra

B-ring : a triple (A, μ, η) M. Takeuchi, G. Böhm ....

A a B-bimodule with B-bimodule maps $\mu : A \otimes_B A \to A$ and $\eta : B \to A$

associativity and unit conditions:

$$\mu \circ (\mu \otimes_B \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes_B \mu), \quad \mu \circ (\eta \otimes_B \mathrm{id}_A) = \mathrm{id}_A = \mu \circ (\mathrm{id}_A \otimes_B \eta).$$

Dually, B-coring : a triple (C, Δ, ε)

C is a B-bimodule with B-bimodule maps $\Delta : C \to C \otimes_B C$ and $\varepsilon : C \to B$

coassociativity and counit conditions:

$$(\Delta \otimes_B \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes_B \Delta) \circ \Delta, \quad (\varepsilon \otimes_B \mathrm{id}_C) \circ \Delta = \mathrm{id}_C = (\mathrm{id}_C \otimes_B \varepsilon) \circ \Delta$$

A left B-bialgebroid $\mathcal{C}$:

a $(B \otimes B^{op})$-ring and a B-coring structure on $\mathcal{C}$ with compatibility conditions

There are source and target maps (with commuting ranges)

$$s := \eta(\,\cdot\, \otimes_B 1_B) : B \to \mathcal{C} \quad \text{and} \quad t := \eta(1_B \otimes_B \,\cdot\,) : B^{op} \to \mathcal{C}$$

The compatibility conditions for a left B-bialgebroid $\mathcal{C}$

(i) The bimodule structures in the B-coring $(\mathcal{C}, \Delta, \varepsilon)$ and those of the $B \otimes B^{op}$-ring $(\mathcal{C}, s, t)$ are related as

$$b \triangleright a \triangleleft \tilde{b} := s(b)t(\tilde{b})a \qquad \text{for } b, \tilde{b} \in B,\ a \in \mathcal{C}.$$

(ii) The coproduct Δ corestricts to an algebra map from $\mathcal{C}$ to

$$\mathcal{C} \times_B \mathcal{C} := \left\{ \sum\nolimits_j a_j \otimes_B \tilde{a}_j \mid \sum\nolimits_j a_j t(b) \otimes_B \tilde{a}_j = \sum\nolimits_j a_j \otimes_B \tilde{a}_j s(b),\ \forall\, b \in B \right\},$$

(iii) The counit $\varepsilon : \mathcal{C} \to B$ satisfies the properties,

(1) $\varepsilon(1_{\mathcal{C}}) = 1_B$,

(2) $\varepsilon(s(b)a) = b\varepsilon(a)$,

(3) $\varepsilon(as(\varepsilon(\tilde{a}))) = \varepsilon(a\tilde{a}) = \varepsilon(at(\varepsilon(\tilde{a})))$, for all $b \in B$ and $a, \tilde{a} \in \mathcal{C}$.

A Hopf algebroid with invertible antipode G. Böhm

For a left bialgebroid $(\mathcal{C}, \Delta, \varepsilon, s, t)$ over the algebra B, an invertible antipode $S : \mathcal{C} \to \mathcal{C}$ in an algebra anti-homomorphism with inverse $S^{-1} : \mathcal{C} \to \mathcal{C}$ s.t.

$$S \circ t = s$$

and compatibility conditions with the coproduct:

$$(Sh_{(1)})_{(1')} h_{(2)} \otimes_B S(h_{(1)})_{(2')} = 1_{\mathcal{C}} \otimes_B Sh$$

$$(S^{-1}h_{(2)})_{(1')} \otimes_B (S^{-1}h_{(2)})_{(2')} h_{(1)} = S^{-1}h \otimes_B 1_{\mathcal{C}}$$

These then imply $S(h_{(1)})\, h_{(2)} = t \circ \varepsilon \circ Sh$.

The above similar to a Hopf algebra with an algebra B as the ground field.

source of difficulties/interest : there is no unique antipode in general

A weaker condition P. Schauenburg

A bialgebroid $\mathcal{C}$ is a Hopf algebroid if the map

$$\lambda : \mathcal{C} \otimes_{B^{op}} \mathcal{C} \to \mathcal{C} \otimes_B \mathcal{C}, \qquad \lambda(p \otimes_{B^{op}} q) = p_{(1)} \otimes_B p_{(2)} q$$

is invertible

$$\otimes_{B^{op}} \quad pt(b) \otimes_{B^{op}} q = p \otimes_{B^{op}} t(b) q \qquad \otimes_B \quad t(b) p \otimes_B q = p \otimes_B s(b) q$$

For $B = k$, this reduces to the map

$$\lambda : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}, \qquad p \otimes q \mapsto p_{(1)} \otimes p_{(2)} q$$

which for a usual Hopf algebra with an antipode has inverse

$$p \otimes q \mapsto p_{(1)} \otimes S(p_{(2)}) q$$

Also here, if there is an invertible antipode S as before one constructs an inverse for the map λ ; for $X, Y \in \mathcal{C}$,

$$\lambda^{-1}(X \otimes_B Y) = S^{-1}(S(X)_{(2)}) \otimes_{B^{op}} S(X)_{(1)} Y$$

No claim that S here is unique

The noncommutative gauge bialgebroid aka Ehresmann–Schauenburg

$B = A^{co\,H} \subseteq A$ be a Hopf–Galois extension

right coaction : $\delta(a) = a_{(0)} \otimes a_{(1)}$

translation map : $\tau(h) = h^{<1>} \otimes_B h^{<2>}$

The B-bimodule $\mathcal{C}(A, H)$ of coinvariant elements for the diagonal coaction,

$$(A \otimes A)^{coH} = \{a \otimes \tilde{a} \in A \otimes A\,;\; a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)}\tilde{a}_{(1)} = a \otimes \tilde{a} \otimes 1_H\}$$

is a B-coring with coproduct and counit:

$$\Delta(a \otimes \tilde{a}) = a_{(0)} \otimes \tau(a_{(1)}) \otimes \tilde{a} = a_{(0)} \otimes a_{(1)}{}^{<1>} \otimes_B a_{(1)}{}^{<2>} \otimes \tilde{a},$$

$$\varepsilon(a \otimes \tilde{a}) = a\tilde{a}.$$

One see $\mathcal{C}(A, H)$ is a subalgebra of $A \otimes A^{op}$ and it is indeed a (left) B-bialgebroid

Product $\quad (x \otimes \tilde{x}) \bullet_{\mathcal{C}(A,H)} (y \otimes \tilde{y}) = xy \otimes \tilde{y}\tilde{x}$

Target and source maps $\quad t(b) = 1_A \otimes b \quad$ and $\quad s(b) = b \otimes 1_A$

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The Ehresmann–Schauenburg bialgebroid $\mathcal{C} = \mathcal{C}(A, H)$ of a Hopf–Galois extension is a Hopf algebroid : there is an explicit map

$$\rho : \mathcal{C} \otimes_B \mathcal{C} \to \mathcal{C} \otimes_{B^{op}} \mathcal{C}$$

which is the inverse of the map λ before going in opposite direction

Furthermore, if the Hopf algebra H is coquasitriangular with R matrix (a convolution invertible map) $\mathcal{R} : H \otimes H \to k$ (+ conditions),

there is an antipod: the inverse of the braiding induced by $\mathcal{R}$:

$$\Psi(a \otimes \tilde{a}) = a_{(0)} \otimes \tilde{a}_{(0)} \otimes \mathcal{R}(a_{(1)} \otimes \tilde{a}_{(1)})$$

this is an invertible H-comodule map with inverse

$$\Psi^{-1}(a \otimes \tilde{a}) = a_{(0)} \otimes \tilde{a}_{(0)} \otimes \mathcal{R}^{-1}(a_{(1)} \otimes \tilde{a}_{(1)})$$

both map restrict to the invariant subspace $\mathcal{C}(A, H)$.

Then $S = \Psi^{-1}$ obeys all properties of an antipode for $\mathcal{C}(A, H)$.

The bialgebroid $\mathcal{C}(A,H)$ of a Hopf–Galois extension as a quantization (of the dualization) of the classical gauge groupoid principal bundle

Its bisections correspond to gauge transformations

$\mathcal{C}(A,H)$ the gauge bialgebroid of a Hopf–Galois extension $B = A^{coH} \subseteq A$

A bisection is a B-bilinear unital left character on the B-ring $(\mathcal{C}(A,H), s)$.

A map $\sigma : \mathcal{C}(A,H) \to B$ such that:

$\sigma(1_A \otimes 1_A) = 1_B$, unitality,

$\sigma\big(s(b)t(\tilde{b})(x \otimes \tilde{x})\big) = b\sigma(x \otimes \tilde{x})\tilde{b}$, B-bilinearity,

$\sigma\big((x \otimes \tilde{x})\, s(\sigma(y \otimes \tilde{y}))\big) = \sigma\big((x \otimes \tilde{x})(y \otimes \tilde{y})\big)$, associativity.

The collection $\mathcal{B}(\mathcal{C}(A,H))$ of bisections of the bialgebroid $\mathcal{C}(A,H)$ is a group with convolution product :

$$\sigma_1 * \sigma_2(x \otimes \tilde{x}) := \sigma_1((x \otimes \tilde{x})_{(1)})\, \sigma_2((x \otimes \tilde{x})_{(2)}) = \sigma_1(x_{(0)} \otimes {x_{(1)}}^{<1>})\, \sigma_2({x_{(1)}}^{<2>} \otimes \tilde{x})$$

using the B-coring coproduct $\Delta(x \otimes \tilde{x}) = (x \otimes \tilde{x})_{(1)} \otimes_B (x \otimes \tilde{x})_{(2)}$

A group isomorphism

$$\alpha : \mathsf{Aut}_H(A) \to \mathcal{B}(\mathcal{C}(A,H))$$

between gauge transformations and bisections:

$$\mathcal{B}(\mathcal{C}(A,H)) \ni \sigma \quad \mapsto \quad F_\sigma(a) := \sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})\, a_{(1)}{}^{<2>}, \quad F_\sigma \in \mathsf{Aut}_H(A)$$

$$F \in \mathsf{Aut}_H(A) \ni F \quad \mapsto \quad \sigma_F(a \otimes \tilde{a}) := F(a)\tilde{a}, \quad \sigma_F \in \mathcal{B}(\mathcal{C}(A,H))$$

Bisection can be given for any bialgebroid

For the general case one would need additional requirements so to get a proper composition law for bisections

Explicit examples

the monopole bundles over the quantum S^2_q

a not faithfully flat example from SL(2)

the SU(2) - bundle $S^7_\theta \to S^4_\theta$

the $SO_\theta(2n)$ bundle $SO_\theta(2n+1) \to S^{2n}_\theta$

some example from q-geometry

change from automorphisms to derivations
(infinitesimal gauge transformations)

Lie algebras of suitable ‘bisections’

braided versions of them

Atiyah sequences of braided Lie algebras of derivations

Braiding then

K a Hopf algebra

K-equivariant H-Hopf–Galois extension $B \subseteq A^H$:

A carries a left action $\triangleright : K \otimes A \to A$ of K, compatible with the H-coaction:

$$(k \triangleright a)_{(0)} \otimes (k \triangleright a)_{(1)} = k \triangleright (a_{(0)} \otimes a_{(1)}) \ .$$

Recall: K is quasitriangular if there exists an invertible element $\mathsf{R} \in K \otimes K$ with respect to which the coproduct Δ of K is quasi-cocommutative

$$\Delta^{cop}(k) = \mathsf{R}\Delta(k)\overline{\mathsf{R}} \qquad \Delta^{cop} := \tau \circ \Delta$$

and $\overline{\mathsf{R}} \in K \otimes K$ the inverse of R, $\mathsf{R}\overline{\mathsf{R}} = \overline{\mathsf{R}}\mathsf{R} = 1 \otimes 1$.

R is required to satisfy,

$$(\Delta \otimes \mathrm{id})\mathsf{R} = \mathsf{R}_{13}\mathsf{R}_{23} \qquad \text{and} \qquad (\mathrm{id} \otimes \Delta)\mathsf{R} = \mathsf{R}_{13}\mathsf{R}_{12}.$$

The Hopf algebra K is triangular when $\overline{\mathsf{R}} = \mathsf{R}_{21} = \tau(\mathsf{R})$, τ the flip.

We further assume the Hopf algebra K to be triangular.

This allows for the study of braided Lie algebras.

A braided Lie algebra associated with a triangular Hopf algebra (K, R), is a K-module $\mathfrak{g}$ with a bilinear map

$$[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

that satisfies the following conditions.

(i) K-equivariance: for $\Delta(k) = k_{(1)} \otimes k_{(2)}$ the coproduct of K,

$$k \triangleright [u, v] = [k_{(1)} \triangleright u, k_{(2)} \triangleright v]$$

(ii) braided antisymmetry:

$$[u, v] = -[\mathsf{R}_\alpha \triangleright v, \mathsf{R}^\alpha \triangleright u],$$

(iii) braided Jacobi identity:

$$[u, [v, w]] = [[u, v], w] + [\mathsf{R}_\alpha \triangleright v, [\mathsf{R}^\alpha \triangleright u, w]]$$

Any K-module algebra A is a K-braided Lie algebra with braided commutator

$$[\ ,\ ] : A \otimes A \to A, \qquad a \otimes b \mapsto [a,b] = ab - (\mathsf{R}_\alpha \triangleright b)\,(\mathsf{R}^\alpha \triangleright a)\,.$$

Also,the K-module algebra $(\mathsf{Hom}(A,A), \triangleright_{\mathsf{Hom}(A,A)})$

$$\triangleright_{\mathsf{Hom}(A,A)}: K \otimes \mathsf{Hom}(A,A) \to \mathsf{Hom}(A,A)$$
$$k \otimes \psi \mapsto k \triangleright_{\mathsf{Hom}(A,A)} \psi :\ A \mapsto k_{(1)} \triangleright_A \psi(S(k_{(2)}) \triangleright A)$$

is a braided Lie algebra with braided commutator.

And so is its K-submodule of braided derivations of A

$$\mathsf{Der}(A) := \{\psi \in \mathsf{Hom}(A,A)\,|\,\psi(aa') = \psi(a)a' + (\mathsf{R}_\alpha \triangleright a)\,(\mathsf{R}^\alpha \triangleright_{\mathsf{Hom}(A,A)} \psi)(a')\}$$

$$[\ ,\ ] : \mathsf{Der}(A) \otimes \mathsf{Der}(A) \to \mathsf{Der}(A)$$

$$\psi \otimes \lambda \mapsto [\psi, \lambda] := \psi \circ \lambda - (\mathsf{R}_\alpha \triangleright_{\mathsf{Der}(A)} \lambda) \circ (\mathsf{R}^\alpha \triangleright_{\mathsf{Der}(A)} \psi).$$

Infinitesimal gauge transformations

$B = A^{coH} \subseteq A$ a K-equivariant Hopf–Galois extension, for (K, R) triangular.

Inside the braided Lie algebra $\mathrm{Der}(A)$ consider the subspace of braided derivations that are H-comodule maps (H-equivariant),

$$\mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) = \big\{ u \in \mathrm{Hom}(A, A) \mid \delta(u(a)) = u(a_{(0)}) \otimes a_{(1)},$$

$$u(aa') = u(a)a' + (\mathsf{R}_\alpha \triangleright a)(\mathsf{R}^\alpha \triangleright u)(a'), \text{ for all } a, a' \in A \big\}$$

and then those derivations that are vertical,

$$\mathrm{aut}^{\mathsf{R}}_B(A) := \{ u \in \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) \mid u(b) = 0, \text{ for all } b \in B \} .$$

Elements of $\mathrm{aut}^{\mathsf{R}}_B(A)$ are regarded as infinitesimal gauge transformations

of the K-equivariant Hopf–Galois extension $B = A^{coH} \subseteq A$.

Twisting

The constructions survive under a Drinfeld twists

Let K a Hopf algebra. A twist for K is an invertible element $\mathsf{F} \in K \otimes K$ which is unital, $(\varepsilon \otimes \mathrm{id})(\mathsf{F}) = 1 = (\mathrm{id} \otimes \varepsilon)(\mathsf{F})$, and satisfies the twist condition

$$(\mathsf{F} \otimes 1)[(\Delta \otimes \mathrm{id})(\mathsf{F})] = (1 \otimes \mathsf{F})[(\mathrm{id} \otimes \Delta)(\mathsf{F})] \ .$$

For F and its inverse $\overline{\mathsf{F}}$ we write $\mathsf{F} = \mathsf{F}^\alpha \otimes \mathsf{F}_\alpha$ and $\overline{\mathsf{F}} =: \overline{\mathsf{F}}^\alpha \otimes \overline{\mathsf{F}}_\alpha$

The algebra (K, m, η) with coproduct

$$\Delta_\mathsf{F}(k) := \mathsf{F}\Delta(k)\overline{\mathsf{F}} = \mathsf{F}^\alpha k_{(1)}\overline{\mathsf{F}}^\beta \otimes \mathsf{F}_\alpha k_{(2)}\overline{\mathsf{F}}_\beta \ , \qquad k \in K$$

is a bialgebra K_F.

If K is a Hopf algebra, then K_F gets a new antipode $S_\mathsf{F}(k) := \mathsf{u}_\mathsf{F} S(k)\bar{\mathsf{u}}_\mathsf{F}$,

$\mathsf{u}_\mathsf{F} := \mathsf{F}^\alpha S(\mathsf{F}_\alpha)$ with $\bar{\mathsf{u}}_\mathsf{F} = S(\overline{\mathsf{F}}^\alpha)\overline{\mathsf{F}}_\alpha$ its inverse.

(K, R) is quasitriangular such is the twisted bialgebra K_F with R-matrix

$$\mathsf{R}_\mathsf{F} := \mathsf{F}_{21}\, \mathsf{R}\, \overline{\mathsf{F}} = \mathsf{F}_\alpha \mathsf{R}^\beta \overline{\mathsf{F}}^\gamma \otimes \mathsf{F}^\alpha \mathsf{R}_\beta \overline{\mathsf{F}}_\gamma$$

and inverse $\overline{\mathsf{R}}_\mathsf{F} := \mathsf{F}\, \overline{\mathsf{R}}\, \overline{\mathsf{F}}_{21} = \mathsf{F}^\alpha \overline{\mathsf{R}}^\beta \overline{\mathsf{F}}_\gamma \otimes \mathsf{F}_\alpha \overline{\mathsf{R}}_\beta \overline{\mathsf{F}}^\gamma$.

If (K, R) is triangular, so is $(K_\mathsf{F}, \mathsf{R}_\mathsf{F})$.

Any K-module algebra V with left action $\triangleright_V \colon K \otimes V \to V$, is also a K_F-module with the same linear map $\triangleright_V$, now thought as a map $\triangleright_V \colon K_\mathsf{F} \otimes V \to V$.

If A is a K-module algebra, with multiplication m_A and unit η_A, in order for the action $\triangleright_{A_\mathsf{F}}$ to be an algebra map one endows the K_F-module A_F with a new algebra structure:

$$m_{A_\mathsf{F}} : A_\mathsf{F} \otimes_\mathsf{F} A_\mathsf{F} \longrightarrow A_\mathsf{F}, \qquad a \otimes_\mathsf{F} a' \longmapsto a \bullet_\mathsf{F} a' := (\overline{\mathsf{F}}^\alpha \triangleright_A a)\, (\overline{\mathsf{F}}_\alpha \triangleright_A a').$$

and the unit is unchanged

For any K-module algebra map $\psi : A \to A'$, the K_F-module map $\psi_F : A_\mathsf{F} \to A'_\mathsf{F}$ is an algebra map for the deformed products.

Atiyah sequences and their splittings (connections)

A K-equivariant Hopf–Galois extension $B = A^{coH} \subseteq A$

The braided Lie algebra of vertical equivariant derivations

$$\mathrm{aut}_B^{\mathsf{R}}(A) := \{u \in \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) \mid u(b) = 0,\ b \in B\}$$

is a braided Lie subalgebra of equivariant derivations

$$\mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) = \{u \in \mathrm{Der}(A) \mid \delta \circ u = (u \otimes \mathrm{id}) \circ \delta\}\ .$$

Each derivation in $\mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A)$, being H-equivariant, restricts to a derivation on the subalgebra of coinvariant elements $B = A^{coH}$

A sequence of braided Lie algebras $\mathrm{aut}_B^{\mathsf{R}}(A) \to \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) \to \mathrm{Der}^{\mathsf{R}}(B)$

When exact,

$$0 \to \mathrm{aut}_B^{\mathsf{R}}(A) \to \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) \to \mathrm{Der}^{\mathsf{R}}(B) \to 0$$

is a version of the Atiyah sequence of a (commutative) principal fibre bundle.

An H-equivariant splitting of the sequence is a connection on the bundle

Examples from θ-deformations

$$F = e^{\pi i\theta(H_1\otimes H_2 - H_2\otimes H_1)} \qquad [H_1, H_2] = 0$$

$$\mathsf{R_F} = \overline{\mathsf{F}}^2 = e^{-2\pi i\theta(H_1\otimes H_2 - H_2\otimes H_1)}$$

Jordanian twist . κ-Minkowski

$$\mathsf{F} = \exp\left(u\tfrac{\partial}{\partial u}\otimes\sigma\right) \qquad \sigma = \ln\left(1 + \frac{1}{\kappa}P_0\right)$$

$$P_0 = iu\tfrac{\partial}{\partial x^0} \qquad [u\tfrac{\partial}{\partial u}, P_0] = P_0$$

In particular $\mathcal{O}(S^4_\theta)$

with generators b_μ, $\mu = (\mu_1, \mu_2) = (0,0), (\pm 1, 0), (0, \pm 1)$

the weights for the action of H_1, H_2.

Their commutation relations are

$$b_\mu \bullet_\theta b_\nu = \lambda^{2\mu\wedge\nu}\, b_\nu \bullet_\theta b_\mu \qquad \lambda = e^{-\pi i \theta}\ .$$

with sphere relation $\sum_{b_\mu} b_\mu^* \cdot_\theta b_\mu = 1$.

$\mathrm{Der}^{\mathsf{R}_\mathsf{F}}(\mathcal{O}(S^4_\theta))$ is generated as an $\mathcal{O}(S^4_\theta)$-module by operators $\widetilde{H}_\mu$

defined on the algebra generators as

$$\widetilde{H}_\mu(b_\nu) := \delta_{\mu^*\nu} - b_\mu \bullet_\theta b_\nu$$

and extended to the whole algebra $\mathcal{O}(S^4_\theta)$ as braided derivations:

$$\widetilde{H}_\mu(b_\nu \bullet_\theta b_\tau) = \widetilde{H}_\mu(b_\nu) \bullet_\theta b_\tau + \lambda^{2\mu\wedge\nu} b_\nu \bullet_\theta \widetilde{H}_\mu(b_\tau).$$

They verify

$$\widetilde{H}_\mu(\sum_\nu b_\nu^* \bullet_\theta b_\nu) = 0\,, \qquad \sum\nolimits_\mu b_\mu^* \bullet_\theta \widetilde{H}_\mu = 0$$

In the classical limit $\theta = 0$, the derivations $\widetilde{H}_\mu$ reduce to

$$H_\mu = \partial_{\mu^*} - b_\mu \Delta, \qquad \Delta = \sum_\mu b_\mu \partial_\mu$$

the Liouville vector field.
The weights μ are those of the five dimensional representation of $so(5)$.

The bracket in $\mathrm{Der}^{\mathsf{R_F}}(\mathcal{O}(S^4_\theta))$ is the braided commutator

$$\begin{aligned}[\widetilde{H}_\mu, \widetilde{H}_\nu]_{\mathsf{R_F}} &:= \widetilde{H}_\mu \circ \widetilde{H}_\nu - \lambda^{2\mu\wedge\nu} \widetilde{H}_\nu \circ \widetilde{H}_\mu \\ &= b_\mu \bullet_\theta \widetilde{H}_\nu - \lambda^{2\mu\wedge\nu} b_\nu \bullet_\theta \widetilde{H}_\mu\end{aligned}$$

The generators $\widetilde{H}_\mu$ can be expressed in terms of their commutators as

$$\widetilde{H}_\nu = \sum_\mu b^*_\mu \bullet_\theta [\widetilde{H}_\mu, \widetilde{H}_\nu]_{\mathsf{R_F}}$$

Denote $$\widetilde{H}^\pi_{\mu,\nu} := [\widetilde{H}_\mu, \widetilde{H}_\nu]_{\mathsf{R_F}} = -\lambda^{2\mu\wedge\nu} \widetilde{H}^\pi_{\nu,\mu}$$

Their braided commutators close the braided Lie algebra $so_\theta(5)$:

$$[\widetilde{H}^\pi_{\mu,\nu}, \widetilde{H}^\pi_{\tau,\sigma}]_{\mathsf{R_F}} = \delta_{\nu^*\tau} \widetilde{H}^\pi_{\mu,\sigma} - \lambda^{2\mu\wedge\nu} \delta_{\mu^*\tau} - \lambda^{2\tau\wedge\sigma} (\delta_{\nu^*\sigma} \widetilde{H}^\pi_{\mu,\tau} - \lambda^{2\mu\wedge\nu} \delta_{\sigma^*\mu} \widetilde{H}^\pi_{\nu,\tau})$$

The instanton $\mathcal{O}(SU(2))$ Hopf–Galois extension $\mathcal{O}(S^4_\theta) \subset \mathcal{O}(S^7_\theta)$.

A short exact sequence of braided Lie algebras

$$0 \to \mathrm{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta)) \xrightarrow{\imath} \mathrm{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta)) \xrightarrow{\pi} \mathrm{Der}(\mathcal{O}(S^4_\theta)) \to 0$$

$\mathrm{Der}(\mathcal{O}(S^4_\theta))$ generated as before by elements $\widetilde{H}^\pi_{\mu,\nu}$

$\mathrm{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta))$ generated by (explicit) derivations $\widetilde{H}_{\mu,\nu}$ realising a representation of $so_\theta(5)$ as derivations on $\mathcal{O}(S^7_\theta)$ and

$$\pi(\widetilde{H}_{\mu,\nu}) = \widetilde{H}^\pi_{\mu,\nu}.$$

$\mathrm{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$ vertical and equivariant (alternatively via a connection)

The horizontal lift: the $\mathcal{O}(S^4_\theta)$-module map $\rho : \mathrm{Der}(\mathcal{O}(S^4_\theta)) \to \mathrm{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta))$ defined on the generators $\widetilde{H}_\nu$ of $\mathrm{Der}^{\mathsf{R}_\mathsf{F}}(\mathcal{O}(S^4_\theta))$ as

$$\rho(\widetilde{H}_\nu) := \sum_\mu b^*_\mu \bullet_\theta \widetilde{H}_{\mu,\nu}$$

is a splitting of the sequence above .

The corresponding vertical projection is the $\mathcal{O}(S^4_\theta)$-module map

$\Psi : \mathrm{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta))) \to \mathrm{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta)$

$$\Psi(\widetilde{H}_{\mu,\nu}) := \widetilde{H}_{\mu,\nu} - \rho(\widetilde{H}^\pi_{\mu,\nu}) = \widetilde{H}_{\mu,\nu} - \big(b_\mu \bullet_\theta \rho(\widetilde{H}_\nu) - \lambda^{2\mu\wedge\nu}\, b_\nu \bullet_\theta \rho(\widetilde{H}_\mu)\big)$$

These derivations generated the algebra $\mathrm{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$.

The curvature

$$\Omega(X,Y) := [\rho(X),\rho(Y)]_{\mathsf{R_F}} - \rho([X,Y]_{\mathsf{R_F}}) = \imath \circ \Psi[\rho(X),\rho(Y)]_{\mathsf{R_F}}$$

One finds $\qquad [\rho(\widetilde{H}_\mu),\rho(\widetilde{H}_\nu)]_{\mathsf{R_F}} = \widetilde{H}_{\mu,\nu}$

Then

$$\Omega(\widetilde{H}_\mu,\widetilde{H}_\nu) = \widetilde{H}_{\mu,\nu} - \big(b_\mu \bullet_\theta \rho(\widetilde{H}_\nu) - \lambda^{2\mu\wedge\nu}\, b_\nu \bullet_\theta \rho(\widetilde{H}_\mu)\big) = \imath \circ \Psi(\widetilde{H}_{\mu,\nu}).$$

There is also a connection 1-form.

The general construction

(K, R) a triangular Hopf algebra ; an exact sequence of K-braided Lie algebras

$$0 \to \mathfrak{g} \xrightarrow{\imath} P \xrightarrow{\pi} T \to 0$$

For B an algebra; take (B, T) a braided Lie–Rinehart pair:

T is a B-module with a braided Lie algebra morphism $T \to \mathsf{Der}^{\mathsf{R}}(B)$;

B is a T-module and T acts as braided derivations of B,

$$X(bb') = X(b)b' + (\mathsf{R}_\alpha \triangleright b)(\mathsf{R}^\alpha \triangleright X)(b'), \qquad b, b' \in B, \quad X \in T,$$

and

$$[X, bX']_{\mathsf{R}} = X(b)X' + (\mathsf{R}_\alpha \triangleright b)[(\mathsf{R}^\alpha \triangleright X), X']_{\mathsf{R}}, \qquad b \in B, \quad X, X' \in T.$$

A connection on the sequence is a splitting: a B-module map,

$$\rho : T \to P, \qquad \pi \circ \rho = \mathsf{id}_T$$

the 'vertical projection', is the B-module map $\omega_\rho : P \to \mathfrak{g}$,

$$\omega_\rho(Y) = Y - \rho(Y^\pi), \qquad Y \in P$$

The extend to which ρ or ω_ρ fail to be braided Lie algebra morphisms is measured by the *(basic)* curvature

$$\Omega(X, X') := \rho([X, X']_{\mathsf{R}}) - [\rho(X), \rho(X')]_{\mathsf{R}}\,, \qquad X, X' \in T.$$

Ω is a $\mathfrak{g}$-valued braided two-form on T.

The curvature can also be given as a basic $\mathfrak{g}$-valued braided two-form on P (*spatial* curvature):

$$\Omega_{\omega_\rho}(Y, Y') := \Omega(Y^\pi, Y'^\pi), \qquad Y, Y' \in P.$$

$$\Omega_{\omega_\rho}(Y, Y') = [Y, \omega_\rho(Y')]_{\mathsf{R}} + [\omega_\rho(Y), Y']_{\mathsf{R}} - \omega_\rho([Y, Y']_{\mathsf{R}}) - [\omega_\rho(Y), \omega_\rho(Y')]_{\mathsf{R}}.$$

This expression can be read as a *structure equation*:

$$d\omega_\rho = \Omega_{\omega_\rho} + [\omega_\rho, \omega_\rho]_{\mathsf{R}}\ .$$

Here

$$d\zeta(Y, Y') := [Y, \zeta(Y')]_{\mathsf{R}} + [\zeta(Y), Y']_{\mathsf{R}} - \zeta([Y, Y']_{\mathsf{R}}), \qquad Y, Y' \in P.$$

(generalised to higher forms)

There is a Bianchi identity:

$$d\Omega_{\omega_\rho} = -[\Omega_{\omega_\rho}, \omega_\rho]_{\mathsf{R}}\ .$$

An R-symmetric map of degree q

$$\varphi : \mathfrak{g} \otimes^{\mathsf{R}} \ldots \otimes^{\mathsf{R}} \mathfrak{g} \to B$$

which intertwining the representation $\mathsf{ad}_{\mathsf{R}} \otimes^{\mathsf{R}} \ldots \otimes^{\mathsf{R}} \mathsf{ad}_{\mathsf{R}}$ of P on $\mathfrak{g} \otimes^{\mathsf{R}} \ldots \otimes^{\mathsf{R}} \mathfrak{g}$ with the action of P on B (ad_{R} is the braided commutator).

α the braided anti-symmetrization.

Then

$$\varphi_\rho = \alpha \circ f(\Omega \otimes^{\mathsf{R}} \ldots \otimes^{\mathsf{R}} \Omega)$$

is a braided B-valued $2q$-form on T.

One has:

$$d\varphi_\rho = 0$$

For the cohomology classes:

$$[\varphi_\rho] = [\varphi_{\rho'}] \qquad \rho, \rho' \quad \text{two connections on the sequence}$$

$$\varphi_\rho = \varphi_{\rho'} + d(.....)$$

Consider:

$\text{Inv}^q = \{$ all such φ as before $\}$ $\qquad \text{Inv} = \oplus_q \text{Inv}^q$

H_{Ch} Chevalley cohomology of (T, B)

we get a linear map

$$\text{cw} : \text{Inv} \to H_{Ch} \qquad \varphi \to [\varphi_\rho]$$

When pulled back to P:

$$\pi^* \varphi_\rho = d(\text{ Chern Simons })$$

Galois objects

of a Hopf algebra H (noncommutative principal bundle over a point)

An H-Hopf–Galois extension A of the ground field $\mathbb{C}$.

Examples:

Group Hopf algebras $H = \mathbb{C}[G]$: equivalence classes of $\mathbb{C}[G]$-Galois objects are in bijective correspondence with the cohomology group $H^2(G, \mathbb{C}^\times)$

$H^2(\mathbb{Z}^r, \mathbb{C}^\times) = (\mathbb{C}^\times)^{r(r-1)/2}$: infinitely many iso classes of $\mathbb{C}[\mathbb{Z}^r]$-Galois objects

Taft algebras : q a primitive N-th root of unity; T_N, neither commutative nor cocommutative Hopf algebra; generators x, g with relations:

$$x^N = 0\,, \quad g^N = 1\,, \quad xg - q\,gx = 0\,.$$

coproduct: $\Delta(x) := 1 \otimes x + x \otimes g, \qquad \Delta(g) := g \otimes g$

counit: $\varepsilon(x) := 0, \varepsilon(g) := 1$, and antipode: $S(x) := -xg^{-1}, S(g) := g^{-1}$.

Equivalence classes of T_N-Galois objects in bijective correspondence with the abelian group $\mathbb{C}$ A. Masuoka

For $s \in \mathbb{C}$, let A_s be the algebra generated by elements X, G with relations:

$$X^N = s\,, \quad G^N = 1\,, \quad XG - q\,GX = 0\,.$$

The algebra A_s is a right T_N-comodule algebra, with coaction defined by

$$\delta^A(X) := 1 \otimes x + X \otimes g, \qquad \delta^A(G) := G \otimes g.$$

The algebra of corresponding coinvariants is just the ground field $\mathbb{C}$.

Thus A_s is a T_N-Galois object

The gauge bialgebroid of a Galois object is a Hopf algebra P. Schauenburg

The coproduct : $\Delta_{\mathcal{C}}(a \otimes \tilde{a}) = a_{(0)} \otimes a_{(1)}{}^{<1>} \otimes a_{(1)}{}^{<2>} \otimes \tilde{a}$,

Counit : $\varepsilon_{\mathcal{C}}(a \otimes \tilde{a}) = a\tilde{a} \in \mathbb{C}$

Antipode : $S_{\mathcal{C}}(a \otimes \tilde{a}) := \tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>} a \tilde{a}_{(1)}{}^{<2>}$

Corollary:

Bisections are characters of the Hopf algebra $\mathcal{C}(A, H)$ with product as before and inverse $\sigma^{-1} = \sigma \circ S_{\mathcal{C}}$, as it is the case for characters

Thus for the gauge group:

$$\mathrm{Aut}_H(A) \simeq \mathcal{B}(\mathcal{C}(A, H)) = \mathrm{Char}(\mathcal{C}(A, H))$$

The Taft algebra :

For any $s \in \mathbb{C}$ there is a Hopf algebra isomorphism

$$\Phi : \mathcal{C}(A_s, T_N) \simeq T_N.$$

The elements

$$\Xi = X \otimes G^{-1} - 1 \otimes XG^{-1}, \qquad \Gamma = G \otimes G^{-1}$$

are coinvariants for the right diagonal coaction of T_N on $A_s \otimes A_s$ and generate $\mathcal{C}(A_s, T_N) = (A_s \otimes A_s)^{co\, T_N}$. They satisfy the Taft algebra relations:

$$\Xi^N = 0, \quad \Gamma^N = 1, \quad \Xi \bullet_{\mathcal{C}} \Gamma - q\, \Xi \bullet_{\mathcal{C}} \Gamma = 0$$

Thus

$$\mathrm{Aut}_{T_N}(A_s) \simeq \mathcal{B}(\mathcal{C}(A_s, T_N)) = \mathrm{Char}(T_N) = \mathbb{Z}_N.$$

Summing up:

Worked out a gauge algebroid for a noncommutative principal bundle

A suitable class of (infinitesimal) gauge transformations

Infinite dimensional Hopf algebra (of possibly braided derivations)

A Chern-Weil homomorphisms and characteristic classes

Chern-Simons terms

some natural structures but we are only at the beginning ...

Thank you