Around heat kernel

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Why computation of spectral action is difficult ?

Given $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, Chamseddine & Connes proposed in 1996:

$$\boxed{S(\mathcal{D}, f) := \operatorname{Tr} f(|\mathcal{D}|)}$$

$f : \mathbb{R}^+ \to \mathbb{R}^+$ positive function

Possible answers

Only a few general results available
Computation involves meromorphic extension and ζ-function
Exact computation impossible since spectrum is unknown
Only asymptotics computations

Consequently appearance of a general scheme:

Laplace transform and heat kernel

If $f(x) = \int_0^\infty e^{-tx}\, \mathrm{d}\phi(t)$, then

$$\operatorname{Tr} f(|\mathcal{D}|) = \int_0^\infty \operatorname{Tr} e^{-t\,|\mathcal{D}|}\, \mathrm{d}\phi(t)$$

Why computation of spectral action is difficult ?

The heat trace

$$t \to \operatorname{Tr} e^{-t\,|\mathcal{D}|}$$

expected

ϕ-integrable at $t \to \infty$ since $|\mathcal{D}| \geq 0$

divergencies at $t = 0$ like $\operatorname{Tr} e^{-t\,|\mathcal{D}|} = \mathcal{O}(t^{-p})$ when $t \to 0$ for some $p > 0$

You can expect

$$\operatorname{Tr} e^{-t|\mathcal{D}|} \underset{t\to 0}{\sim} t^{-p} \sum_{r=0}^{\infty} c_r(\mathcal{D})\,(t^{\alpha})^r \quad \text{with } \alpha \geq 0$$

Control of spectral action needs knowledge of : $t \geq 0 \to e^{-t\,|\mathcal{D}|}$

World of semigroup

Semigroups with generators

If $A \in \mathcal{B}(\mathcal{H})$, then $U_t = e^{-tA} := \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} A^n$ gives

$$U_0 = \mathbb{1}$$
$$U_{t+t'} = U_t U_{t'} = U_{t'} U_t \qquad t \geq 0,\ t' \geq 0$$

If A is unbounded the series is ill-defined

Semigroup

Problem: When a semigroup $\{U_t\}_{t\geq 0}$ has a generator ?
Answer: (Hille–Yosida)
On a Banach space, if $\{U_t\}_{t\geq 0}$ is a contraction semigroup ($\|U_t\| \leq 1$, $\forall t \geq 0$), then there exists a unique, closed, densely defined operator A s. t.

$$U_t = e^{-t\,A}$$
$$i.e:\ \partial_t(U_t\,\psi) = -AU_t\,\psi = -U_tA\,\psi, \quad \forall \psi \in \mathrm{Dom}(A)$$
$$U_t = \lim_{n\to\infty}(1 + \tfrac{t}{n}A)^{-n} \quad \text{Euler formula}$$
$$(-\infty, 0) \subset \rho(A)$$
$$\|(A + \lambda\mathbb{1})^{-1}\| \leq \lambda^{-1} \quad \forall\lambda > 0$$

Idea: $A = s\text{-}\lim_{t\to 0} \frac{1}{t}(\mathbb{1} - U_t)$

Semigroup

A strongly continuous semigroup $\{(U_t\}_{t\geq 0}$ of bounded operators on a Banach space has a generator: $U_t = e^{-t\,A}$

If A is the generator on a strongly continuous semigroup $\{U_t\}_{t\geq 0}$, then

$$t \geq 0 \to e^{-t\,A}\psi \in \mathcal{H} \quad \text{is differentiable } \forall\psi \in \mathrm{Dom}(A)$$

Traps:

If differentiable for all $\psi \in \mathcal{H}$ then A is bounded
Even with A unbounded, $t > 0 \to e^{-tA}\psi$ is differentiable $\forall\psi \in \mathcal{H}$

Gibbs semigroup:
strongly continuous semigroup $\{U_t\}_{t\geq 0}$ with $U_t \in \mathcal{L}^1$ for $t > 0$

Schatten class: $\mathcal{L}^p$ endowed with $\|\cdot\|_p$, $p \geq 1$

Semigroup

Let A be a densely defined closed operator on a separable Hilbert space $\mathcal{H}$
A^*A is a positive selfadjoint operator such that $\mathrm{Dom}(A^*A)$ is a core for A
and $0 \leq (\mathbb{1} + A^*A)^{-1} \leq 1$

Lemma

Let $p \geq 1$
(i) If $(\mathbb{1} + A^*A)^{-1} \in \mathcal{L}^p$, then $\{e^{-t\,A^*A}\}_{t\geq 0}$ is a Gibbs semigroup and
$\|e^{-t\,A^*A}\|_1 = \mathrm{Tr}\; e^{-t\,A^*A} \underset{t\to 0}{=} \mathcal{O}(t^{-p})$
(ii) Conversely, suppose $\{e^{-t\,A^*A}\}_{t\geq 0}$ is a Gibbs semigroup and
$\|e^{-t\,A^*A}\|_1 \underset{t\to 0}{=} \mathcal{O}(t^{-p})$. *Then, $(\mathbb{1} + A^*A)^{-1} \in \mathcal{L}^q$ for any $q > p$*

Remark: $\|e^{-t\,A^*A}\|_1 \underset{t\to 0}{\sim} t^{-p} \not\Rightarrow (\mathbb{1} + A^*A)^{-1} \in \mathcal{L}^p$

Semigroup

Remarks:

- Possible generalization of previous result when A is not closed via Krein-von Neumann and Friedrich extensions

- When $(\mathbb{1} + A^*A)^{-1} \in \mathcal{L}^p$ and $(\mathbb{1} + A^*A)^{-1} \notin \mathcal{L}^{p-\epsilon}$ for any $\epsilon > 0$, the zeta function

$$\zeta(s) := \operatorname{Tr}(\mathbb{1} + A^*A)^{-s}$$

is defined for $s \in \mathbb{C}$ with $\Re s > p$
One can expect a pole of the meromorphic extension (if existing) of ζ located at $s = p$

Semigroup in NCG

$(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is
p-summable when $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^p$ (so $\mathcal{D}^{-1} \in \mathcal{L}^p$)
θ-summable when $\operatorname{Tr} e^{-t\,\mathcal{D}^2} < \infty$ for $t > 0$ (i.e. $\{e^{-t\,\mathcal{D}^2}\}_{t\geq 0}$ is Gibbs)

Corollary

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a θ-summable spectral triple.
Then it is p-summable if and only if $\| e^{-t\,\mathcal{D}^2} \|_1 \underset{t\to 0}{=} \mathcal{O}(t^{-p/2})$

θ-summability is quite restricting:

$(\mathcal{A} = l^\infty(\mathbb{N}), \mathcal{H} = l^2(\mathbb{N}), \mathcal{D})$ with $\mathcal{D}$ be a diagonal

If $\sigma(\mathcal{D}) = \{[\log(n+1)]^{1/2} \mid n \in \mathbb{N}\}$, then $\operatorname{Tr} e^{-t\,\mathcal{D}^2} = \sum_{n=0}^{\infty} (n+1)^{-t} < \infty$ only for $t > 1$ even if $e^{-t\,\mathcal{D}^2}$ is a compact $\forall t > 0$

If $\sigma(\mathcal{D}) = \{[\log\log(n+3)]^{1/2} \mid n \in \mathbb{N}\}$, then $\operatorname{Tr} e^{-t\,\mathcal{D}^2} = \sum_{n=0}^{\infty} [\log(n+3)]^{-t}$ is never finite $\forall t > 0$.

Generalization to arbitrary semigroup

Problem: Characterize generators with a given asymptotics:

$$\|e^{-tA}\|_1 \underset{t\to 0}{=} \mathcal{O}(t^{-p}), \qquad \text{for some } p$$

Even if A not selfadjoint

Idea:
Cover the case of a differential operator acting on a fiber bundle over a manifold
Also in a pre-spectral triple (Connes-Levitina-McDonald-Sukochev-Zanin) $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, D is not selfadjoint
Important for manifolds with boundaries

Generalization to arbitrary semigroup

Theorem (I.-Zagrebnov)

Let $\{e^{-tA}\}_{t\geq 0}$ be a strongly continuous semigroup on $\mathcal{H}$ and $p \geq 1$
Then the following are equivalent
(a) $\{e^{-tA}\}_{t\geq 0}$ is Gibbs semigroup with asymptotic $\|e^{-tA}\|_1 \underset{t\to 0}{=} \mathcal{O}(t^{-p})$
(b) For any $q > p$,
(i) the map: $t > 0 \to e^{-tA}$ is $\|\cdot\|_q$-continuous
(ii) $(A - z\mathbb{1})^{-1} \in \mathcal{L}^q(\mathcal{H})$ for $z \in \rho(A)$

$(A - z\mathbb{1})^{-1}$: Laplace's transform of $\{e^{-tA}\}_{t\geq 0}$ via a $\|\cdot\|_q$-Bochner integral

N.B. Assumtion (i) does not imply that $e^{-tA} \in \mathcal{L}^q(\mathcal{H})$
but only $\|e^{-tA} - e^{-sA}\|_q \to 0$ when $t \to s$
neither that this semigroup is holomorphic

Revisiting computation for a differential operator

(M, g): compact boundaryless d-dimensional smooth oriented Riemannian manifold
V: smooth hermitean vector bundle over M of fiber $\mathbb{C}^N$
P: strongly elliptic linear differential operator of degree p with smooth coefficients

Problem: compute the coefficient of the asymptotics

$$\mathrm{Tr}\,(b\, e^{-tP}) \underset{t\downarrow 0}{\sim} t^{-d/p} \sum_{r=0}^{\infty} a_r(b, P)\,(t^{1/p})^r$$

$b \in C^\infty(M, \mathrm{End}\,(\mathrm{V}))$
Strong ellipticity: if P_p be the principal symbol of P, there exists $c > 0$

$$\Re(\langle P_p(\mathbf{x}, \xi)\, v,\ v\rangle_{V_\mathbf{x}}) \geq c\,\|\xi\|^p\,\|v\|^2,$$
$$\forall\,(\mathbf{x}, \xi) \in M \times T^*_\mathbf{x} M,\ \xi \neq 0,\ v \in \mathbb{C}^N$$

Revisiting computation for a differential operator

Warning:
Strong elliptcty implies that p is even
The principal symbol $P_p(\mathbf{x}, \xi)$ is not a scalar or a strictly positive definite matrix, P not symmetric a priori

Several results:
Avramidi, Branson, Gilkey, Pierzchalski, Fulling, Gusynin, Gorbar, Kornyak, Ananthanarayan, Moss, Toms

In NCG:
Connes, Tretkoff, Fathizadeh, Khalkhali, Lesch, Sukochev, Zanin, Dabrowski, Sitarz, Ponge, Ha, Lee, McDonald, Liu, Wang & Wang, ...

Computation for a differential operator

$\mathcal{H} = L^2(M, V)$ for

$$\langle \mathscr{s}, \mathscr{s}' \rangle := \int_M \mathrm{dvol}_g(\mathbf{x}) \, \langle \mathscr{s}(\mathbf{x}), \mathscr{s}'(\mathbf{x}) \rangle_{V_{\mathbf{x}}}$$

Sobolev spaces:
$\mathcal{H}_s(M, V)$, $s \in \mathbb{R}$ with $\|\cdot\|_s$

Results:
$P : \mathcal{H}_{s+p} \to \mathcal{H}_s$ is continuous so bounded and extension still denoted P
P has a discrete spectrum in an angular sector symmetric around $\mathbb{R}$
$\{e^{-t\,P}\}_{t \geq 0}$ is a Gibbs semigroup (with an holomorphic extension $\{e^{-z\,P}\}$)
Since $e^{-t\,P} = e^{-\frac{t}{2}\,P}\, e^{-\frac{t}{2}\,P}$ is Hilbert-Schmidt, the heat trace has an integral kernel:

$$\mathrm{Tr}_{\mathcal{H}}(b\, e^{-t\,P}) = \int_M \mathrm{dvol}_g(\mathbf{x}) \; \mathrm{tr}_N \left[b(\mathbf{x})\, \mathcal{K}(t, \mathbf{x}, \mathbf{x}) \right]$$

Exponential as a series

Idea of Hille (1948)

Theorem

Given $n \in \mathbb{N}$ *and* $s \geq 0$, *let* $\psi \in \mathcal{H}_{(n+2)p+s}(M, V)$, *if* $(n+2)p + s > 1 + \frac{d}{2}$

$$\| \, e^{-tP} \, \psi - \sum_{k=0}^{n} \frac{(-t)^{\,k}}{k!} \, P^k \, \psi \, \|_\infty \underset{t \downarrow 0}{\sim} 0$$

Localization process

Given an open set U from trivialization of V, take the restriction P_U of P
Need closure of P_U:
Natural choice for avoiding the boundary of U is the Dirichlet realization $\overline{P_U}$ of P_U

Corollary

For any $\mathfrak{s} \in C_c^\infty(U, V_U)$ *(with extension* $\widetilde{\mathfrak{s}}$ *by zero on* $M \backslash U$

$$\| \, [\, e^{-tP} \tilde{\mathfrak{s}}](x) - [e^{-t\overline{P_U}} \, \mathfrak{s} \,] \, (x) \, \|_{\mathbb{C}^N} \underset{t \downarrow 0}{\sim} 0 \qquad x \in U$$

No need of Pseudodifferential Theory!

A bit of Fourier transform

For $\xi = (\xi_\mu) \in \mathbb{R}^d$

$$\partial_\mu(e^{ix\cdot\xi}\mathscr{s}) = e^{ix\cdot\xi}(\partial_\mu + i\xi_\mu)\,\mathscr{s}$$

so we use the operator

$$\mathcal{P}_{U,\xi}(x,\partial) := P_U(x, \partial + i\xi)$$

is a differential operator on $C_c^\infty(U, V_U)$
modulo some adventures

$$\mathcal{K}_U(t,x,x) = (2\pi)^{-d}|g|^{-1/2}(x)\int_{\mathbb{R}^d} \mathrm{d}\xi\; e^{-t\,\mathcal{P}_U(x,\partial,\xi)}$$

$$\text{(change of variable } \xi \to t^{-1/p}\xi\text{)}$$

$$= t^{-d/p}(2\pi)^{-d}|g|^{-1/2}(x)\int_{\mathbb{R}^d} \mathrm{d}\xi\; e^{-\widehat{\mathcal{P}}(x,\partial,\xi;t)}$$

where

$$\widehat{\mathcal{P}}(x,\partial,\xi;t) := \sum_{\ell=0}^{p} t^{1-\ell/p}\,\mathcal{P}_\ell(x,\partial,\xi)$$

$\mathcal{P}_\ell$: homogeneous polynomials of degree ℓ in ξ of $\mathcal{P}_{U,\xi}$

Duhamel formula and Volterra series

$A,\ B$: operators such that A and $A+B$ are generators of semigroups
Duhamel equality (strong sense)

$$e^{-t(A+B)} = e^{-tA} - \int_0^t \mathrm{d}s_1\, e^{(s_1-t)\,A}\, B\, e^{-s_1\,(A+B)}$$

by iteration

$$e^{-t\,(A+B)} = \sum_{k=0}^{L} (-1)^k \int_{\Delta_k(t)} \mathrm{d}s\, e^{(s_1-t)\,A}\, B\, e^{(s_2-s_1)\,A}\, B \cdots B\, e^{(s_k-s_{k-1})\,A}\, B\, e^{-s_k\,A}$$
$$+ \text{Remainder(L)}$$

$$\Delta_k(t) := \{ s = (s_1, \dots, s_k) \in \mathbb{R}_+^k \mid 0 \le s_k \le s_{k-1} \le \cdots \le s_2 \le s_1 \le t \}$$

Algebraic expansion

Idea: Remainder may disapears by asymptotics

$$e^{-t\,(A+B)} \underset{t\to 0}{\sim} \sum_{k=0}^{\infty} (-1)^k \int_{\Delta_k(t)} \mathrm{d}s\, e^{(s_1-t)\,A}\, B\, e^{(s_2-s_1)\,A}\, B \cdots B\, e^{(s_k-s_{k-1})\,A}\, B\, e^{-s_k\,A}$$

$$= \sum_{k=0}^{\infty} (-1)^k \qquad f_k(A)[B^{\otimes^k}]$$

where more generally

$$f_k(A)[B_1 \otimes \cdots \otimes B_k] := \int_{\Delta_k(t)} \mathrm{d}s\, e^{(s_1-s_0)\,A}\, B_1\, e^{(s_2-s_1)\,A}\, B_2 \cdots B_k\, e^{(s_{k+1}-s_k)\,A}$$

Algebraic expansion

To do:

$$\int_{\mathbb{R}^d} \mathrm{d}\xi \; e^{-\widehat{\mathcal{P}}(x,\partial,\xi;t)}$$

Decompose

$\widehat{\mathcal{P}} = A + B$ with $A = \mathcal{P}_p$ and $B = \sum_{\ell=0}^{p-1} t^{1-\ell/p}\, \mathcal{P}_\ell$

So

$A \in M_N[x,\xi]$ and $B \in M_N[x,\xi;\partial,t]$

Algebraic expansion compatible with derivation

Proposition

Let $\jmath$ *be a local trivialization* $\jmath : U \to \mathbb{C}^N$ *of a section in* $\Gamma(V)$
and $B_1, \ldots, B_k \in M_N[x, \xi, \partial]$
Then

$$f_k(\xi)[B_1 \otimes \cdots \otimes B_i\partial_\nu \otimes \cdots \otimes B_k]\,\jmath =$$

$$\sum_{j=i+1}^{k} f_k(\xi)[B_1 \otimes \cdots \otimes (\partial_\nu B_j) \otimes \cdots \otimes B_k]\,\jmath$$

$$-\sum_{j=i}^{k} f_{k+1}(\xi)[B_1 \otimes \cdots \otimes B_j \otimes (\partial_\nu \mathcal{P}_d) \otimes B_{j+1} \otimes \cdots \otimes B_k]\,\jmath$$

$$+ f_k(\xi)[B_1 \otimes \cdots \otimes B_i \otimes \cdots \otimes B_k](\partial_\nu \jmath)$$

Important: this is compatible with a covariant derivation ∇

En résumé

$$\mathrm{Tr}\,(b\,e^{-tP}) \underset{t\downarrow 0}{\sim} t^{-d/p} \sum_{r=0}^{\infty} a_r(b,P)\,(t^{1/p})^r, \qquad b \in C^\infty(M, \mathrm{End}\,(V))$$

$$a_r(b,P) = \int_M \mathrm{dvol}_g(\mathbf{x})\, a_r(b,P)(\mathbf{x}), \qquad a_r(b,P)(\mathbf{x}) := \mathrm{tr}\,[b(\mathbf{x})\,\mathcal{R}_r(\mathbf{x})]$$

Theorem (I.-Masson)

For the diagonal kernel integral of $\mathrm{Tr}\;b\,e^{-tP}$

$$\mathcal{R}_r = (2\pi)^{-d}|g|^{-1/2} \sum_{k=0}^{r} \int_{\mathbb{R}^d} \mathrm{d}\xi\, f_k[\mathbb{B}_r^k]$$

where $\mathbb{B}_r \in \oplus_{k\geq 0} M_N[\xi,\partial]^{\otimes^k}$ *is defined by the formal series*

$$\sum\nolimits_{r\geq 0} t^{r/p}\,\mathbb{B}_r := \sum_{k\geq 0} (-1)^k \Big(\sum_{\ell=0}^{p-1} t^{1-\ell/p}\,\mathcal{P}_\ell\Big)^{\otimes^k}$$

Example: Laplace–Beltrami type operator

Covariant derivative ∇ on V

$$P = -(|g|^{-1/2}\nabla_\mu \, |g|^{1/2} \, H^{\mu\nu} \, \nabla_\nu + p^\mu \, \nabla_\mu + q)$$

where $|g| := \det(g_{\mu\nu})$ and $H^{\mu\nu},\ p^\mu,\ q$ are sections of $\mathrm{End}\,(V)$

$$P = -(H^{\mu\nu}\nabla_\mu\nabla_\nu + L^\mu\nabla_\mu + q)$$

with

$$L^\mu := p^\mu + (\nabla_\nu H^{\nu\mu}) + \tfrac{1}{2}\partial_\nu \ln(|g|) \, H^{\nu\mu}$$

Strong ellipticity of P:

$$H(\xi; x) := H^{\mu\nu}(x)\, \xi_\mu \xi_\nu > 0 \quad \text{for any } 0 \neq \xi \in \mathbb{R}^d \text{ and any } x \in M$$

Laplace–Beltrami Algebraic part

$\mathcal{A}$: algebra of matrix-valued functions on (the chart of) M, depending on ξ
Define for $a, b_i \in \mathcal{A}$

$$\mathbf{m}(b_1 \otimes \cdots \otimes b_k) := b_1 \cdots b_k$$

$$R_\ell(a)[b_0 \otimes \cdots \otimes b_k] := b_0 \otimes \cdots \otimes b_\ell a \otimes \cdots \otimes b_k$$

$$C_k(s, a) := \sum_{\ell=0}^{k} (s_\ell - s_{\ell+1})\, R_\ell(a) \quad \text{for } s \in \Delta_k$$

$$b_1 \otimes \cdots \otimes b_k := \xi_{\mu_1} \cdots \xi_{\mu_p} \{b_1 \otimes \cdots \otimes b_k\}^{\mu_1 \ldots \mu_p}$$

Define the family of operators for any $k \in \mathbb{N}$, $p \in \mathbb{N}$ and $\mu_\ell \in \{1, \ldots, d\}$ by

$$\mathbf{X}_{k,\mu_1,\ldots,\mu_p} := \tfrac{1}{(2\pi)^d}\, \mathbf{m} \circ \int_{\Delta_k} \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}\xi\, \xi_{\mu_1} \cdots \xi_{\mu_p}\, e^{-C_k(s,\, H(\xi))}$$

Fully mimick previous $\int_{\mathbb{R}^d} \mathrm{d}\xi\, e^{-\widehat{\mathcal{P}}(x,\partial,\xi;t)}$ with Duhamel expansion
Computation is now purely algebraic

Laplace–Beltrami Algebraic part

$K = \xi_\mu\, K^\mu$ with

$$K^\mu := -i\big(L^\mu + 2H^{\mu\nu}\nabla_\nu\big)$$

By previous theorem

$$\mathcal{R}_r = |g|^{-1/2} \sum_{r/2\leq k\leq r} (-1)^k \sum_{\substack{S\subset\{1,\ldots,k\}\\ |S|=2k-r}} \mathbf{X}_{k,\mu_1\ldots\mu_{2k-r}}\big[\{B_1\otimes\cdots\otimes B_k\}^{\mu_1\ldots\mu_{2k-r}}\big]$$

$$\text{with} \quad \begin{cases} B_i = P & \text{if } i\notin S \\ B_i = K & \text{if } i\in S \end{cases}$$

Laplace–Beltrami Algebraic part

$$\mathcal{R}_0(x) = |g|^{-1/2}\,\mathbf{X}_0[1]$$

$$\mathcal{R}_2(x) = |g|^{-1/2}\,(\mathbf{X}_{2,\mu_1\mu_2}[\{K \otimes K\}^{\mu_1\mu_2}] - \mathbf{X}_1[P])$$

$$\mathcal{R}_4(x) =$$

$$|g|^{-1/2}\,\big(\mathbf{X}_2[P \otimes P] - \mathbf{X}_{3,\mu_1\mu_2}[\{K \otimes K \otimes P\}^{\mu_1\mu_2}] - \mathbf{X}_{3,\mu_1\mu_2}[\{K \otimes P \otimes K\}^{\mu_1\mu_2}]$$
$$- \mathbf{X}_{3,\mu_1\mu_2}[\{P \otimes K \otimes K\}^{\mu_1\mu_2}] + \mathbf{X}_{4,\mu_1\mu_2\mu_3\mu_4}[\{K \otimes K \otimes K \otimes K\}^{\mu_1\mu_2\mu_3\mu_4}]\big)$$

As before, this **X**-calculus is compatible

- with ∇

- with contraction over the index μ_i

Laplace–Beltrami Geometric part

Total covariant derivative $\widehat{\nabla}$: combines the (gauge) connection ∇ on V with the Levi-Civita covariant derivative ${}^g\nabla$ induced by the metric g

$$\widehat{\nabla}_\mu u = \nabla_\mu u = \partial_\mu u + [A_\mu, u]$$

$$\widehat{\nabla}_\mu a^\nu = \nabla_\mu a^\nu + \Gamma^\nu_{\mu\rho} a^\rho \qquad \widehat{\nabla}_\mu b_\nu = \nabla_\mu b_\nu - \Gamma^\rho_{\mu\nu} b_\rho$$

u: $(0,0)$-tensor (End (V)-valued)
$a = a^\nu \partial_\nu$: $(1,0)$-tensor
$b = b_\nu \, dx^\nu$: $(0,1)$-tensor
A_μ is the (local) gauge potential associated to ∇

Proposition

There exist a connection $\widehat{\nabla}'$ and a section q' of End (V) such that

$$P = -(H^{\mu\nu}\, \widehat{\nabla}'_\mu \widehat{\nabla}'_\nu + q')$$

given by $\widehat{\nabla}'_\mu := \widehat{\nabla}_\mu + \frac{1}{2} H_{\mu\rho} N^\rho$ *and* $q' := q - \frac{1}{2} H^{\mu\nu}\, \widehat{\nabla}_\mu (H_{\nu\rho} N^\rho) - \frac{1}{4} N^\nu H_{\nu\sigma} N^\sigma$

$$N^\nu := L^\nu + \Gamma^\nu_{\rho\sigma} H^{\rho\sigma}$$

Theorem (I.-Masson)

Section $\mathcal{R}_2$ of End(V) is

$$|g|^{1/2}\mathcal{R}_2 =$$

$$+\mathbf{X}_1[q] - 2\mathbf{X}_{2,\mu_1\mu_2}[H^{\mu_1\nu_1} \otimes H^{\mu_2\nu_2}]\, F_{\nu_1\nu_2} + 2\mathbf{X}_{3,\mu_1\mu_2\mu_3\mu_4}[H^{\mu_1\nu_1} \otimes H^{\mu_2\nu_2} \otimes [F_{\nu_1\nu_2}, H^{\mu_3\mu_4}]]$$
$$+\tfrac{2}{3} R_{\nu_1\nu_2\,\nu_3}{}^{\mu_1}\mathbf{X}_{2,\mu_1\mu_2}[H^{\nu_1\nu_3} \otimes H^{\mu_2\nu_2}] + \tfrac{4}{3} R_{\nu_1\nu_2\,\nu_3}{}^{\mu_1}\mathbf{X}_{2,\mu_1\mu_2}[H^{\mu_2\nu_1} \otimes H^{\nu_2\nu_3}]$$
$$+\tfrac{4}{3} R_{\nu_1\nu_2\,\nu_3}{}^{\mu_1}\mathbf{X}_{3,\mu_1\mu_2\mu_3\mu_4}[H^{\mu_2\nu_1} \otimes H^{\mu_3\nu_2} \otimes H^{\mu_4\nu_3}] - \tfrac{8}{3} R_{\nu_1\nu_2\,\nu_3}{}^{\mu_1}\mathbf{X}_{3,\mu_1\mu_2\mu_3\mu_4}[H^{\mu_2\nu_1} \otimes H^{\mu_3\nu_3} \otimes H^{\mu_4\nu_2}]$$
$$-2\mathbf{X}_{3,\mu_1\mu_2\mu_3\mu_4}[H^{\mu_1\nu_1} \otimes (\widehat{\nabla}^2_{\nu_1\nu_2} H^{\mu_2\mu_3}) \otimes H^{\mu_4\nu_2}] + \mathbf{X}_{2,\mu_1\mu_2}[(\widehat{\nabla}_{\nu_1} H^{\nu_1\mu_1}) \otimes (\widehat{\nabla}_{\nu_2} H^{\nu_2\mu_2})]$$
$$-2\mathbf{X}_{3,\mu_1\mu_2\mu_3\mu_4}[H^{\mu_1\nu_1} \otimes (\widehat{\nabla}_{\nu_1} H^{\mu_2\mu_3}) \otimes (\widehat{\nabla}_{\nu_2} H^{\nu_2\mu_4})] - 2\mathbf{X}_{3,\mu_1\mu_2\mu_3\mu_4}[(\widehat{\nabla}_{\nu_1} H^{\nu_1\mu_1}) \otimes (\widehat{\nabla}_{\nu_2} H^{\mu_2\mu_3}) \otimes H^{\mu_4\nu_2}]$$
$$+4\mathbf{X}_{4,\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}[H^{\mu_1\nu_1} \otimes (\widehat{\nabla}_{\nu_1} H^{\mu_2\mu_3}) \otimes (\widehat{\nabla}_{\nu_2} H^{\mu_4\mu_5}) \otimes H^{\mu_6\nu_2}]$$
$$+\mathbf{X}_{2,\mu_1\mu_2}[p^{\mu_1} \otimes (\widehat{\nabla}_{\nu_1} H^{\nu_1\mu_2})] - 2\mathbf{X}_{3,\mu_1\mu_2\mu_3\mu_4}[p^{\mu_1} \otimes (\widehat{\nabla}_{\nu_1} H^{\mu_2\mu_3}) \otimes H^{\mu_4\nu_1}]$$
$$-\mathbf{X}_{2,\mu_1\mu_2}[(\widehat{\nabla}_{\nu_1} H^{\nu_1\mu_1}) \otimes p^{\mu_2}] + 2\mathbf{X}_{3,\mu_1\mu_2\mu_3\mu_4}[H^{\mu_4\nu_1} \otimes (\widehat{\nabla}_{\nu_1} H^{\mu_1\mu_2}) \otimes p^{\mu_3}]$$
$$-\mathbf{X}_{2,\mu_1\mu_2}[p^{\mu_1} \otimes p^{\mu_2}] - 2\mathbf{X}_{2,\mu_1\mu_2}[H^{\mu_1\nu_1} \otimes (\widehat{\nabla}_{\nu_1} p^{\mu_2})]$$

Corollary

For $H^{\mu\nu} = g^{\mu\nu} u$ *with* $u > 0$

$$
\begin{aligned}
&(4\pi)^{d/2}\mathcal{R}_2 = \\
&+ \tfrac{1}{6}\mathfrak{R} X_{d/2,1}[u] + X_{d/2,1}[q] - \tfrac{1}{2}(d+2) g^{\nu_1\nu_2} X_{d/2+2,3}[u \otimes (\widehat{\nabla}^2_{\nu_1\nu_2} u) \otimes u] \\
&+ \tfrac{1}{2} g^{\nu_1\nu_2} X_{d/2+1,2}[(\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u)] \\
&- \tfrac{1}{2}(d+2) g^{\nu_1\nu_2} X_{d/2+2,3}[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u)] \\
&- \tfrac{1}{2}(d+2) g^{\nu_1\nu_2} X_{d/2+2,3}[(\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] \\
&+ \tfrac{1}{2}(d+4)(d+2) g^{\nu_1\nu_2} X_{d/2+3,4}[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] \\
&- X_{d/2+1,2}[u \otimes (\widehat{\nabla}_{\nu_1} p^{\nu_1})] + \tfrac{1}{2} X_{d/2+1,2}[p^{\nu_1} \otimes (\widehat{\nabla}_{\nu_1} u)] - \tfrac{1}{2} X_{d/2+1,2}[(\widehat{\nabla}_{\nu_1} u) \otimes p^{\nu_1}] \\
&- \tfrac{1}{2}(d+2) X_{d/2+2,3}[p^{\nu_1} \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u] + \tfrac{1}{2}(d+2) X_{d/2+2,3}[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes p^{\nu_1}] \\
&- \tfrac{1}{2} g_{\nu_1\nu_2} X_{d/2+1,2}[p^{\nu_1} \otimes p^{\nu_2}]
\end{aligned}
$$

Corollary

For $H^{\mu\nu} = h^{\mu\nu}\mathbb{1}$ (i.e. $u = \mathbb{1}$), the section $\mathcal{R}_2$ of End(V) is

$$(4\pi)^{d/2}|g|^{1/2}|h|^{-1/2}\mathcal{R}_2 =$$
$$+ \tfrac{1}{6}h^{\nu_1\nu_2}Ric_{\nu_1\nu_2} - \tfrac{1}{6}(\widehat{\nabla}^2_{\nu_1\nu_2}h^{\nu_1\nu_2}) - \tfrac{1}{12}h^{\nu_1\nu_2}h_{\nu_3\nu_4}(\widehat{\nabla}^2_{\nu_1\nu_2}h^{\nu_3\nu_4})$$
$$- \tfrac{1}{12}h_{\nu_1\nu_2}(\widehat{\nabla}_{\nu_3}h^{\nu_1\nu_2})(\widehat{\nabla}_{\nu_4}h^{\nu_4\nu_3}) + \tfrac{1}{12}h_{\nu_1\nu_2}(\widehat{\nabla}_{\nu_3}h^{\nu_4\nu_1})(\widehat{\nabla}_{\nu_4}h^{\nu_3\nu_2})$$
$$+ \tfrac{1}{48}h_{\nu_1\nu_2}h_{\nu_3\nu_4}h^{\nu_5\nu_6}(\widehat{\nabla}_{\nu_5}h^{\nu_1\nu_2})(\widehat{\nabla}_{\nu_6}h^{\nu_3\nu_4}) + \tfrac{1}{24}h_{\nu_1\nu_2}h_{\nu_3\nu_4}h^{\nu_5\nu_6}(\widehat{\nabla}_{\nu_5}h^{\nu_1\nu_3})(\widehat{\nabla}_{\nu_6}h^{\nu_2\nu_4})$$
$$+ q - \tfrac{1}{2}(\widehat{\nabla}_{\nu_1}p^{\nu_1}) - \tfrac{1}{4}h_{\nu_1\nu_2}p^{\nu_1}p^{\nu_2}$$

A lot of other cases: (Yang-Mills theory)

$H^{\mu\nu} := h^{\mu\nu}u + \zeta Q^{\mu\nu}$

metric h on M

$u > 0$

$\zeta \in \mathbb{R}$ and a tensorial section $Q^{\mu\nu}$ of End(V) such that:

  for any $\sigma \in \mathbb{S}^{d-1}_h$, $Q(\sigma) := Q^{\mu\nu}\sigma_\mu\sigma_\nu$ is a projection and $[Q(\sigma), u] = 0$

Interest

No needs of pseudodifferential theory

Purely algebraic method:
Thierry Masson wrote a program from scratch
Lot of relations between the $\mathbf{X}_{n,\mu_1\cdots\mu_k}$, the derivation and contraction with μ_i

This cover the case of the noncommutative torus

When $d = 2$ and $\theta = p/q$, then

$$C(T^2_\theta) \simeq \Gamma(A_\theta)$$

algebra of continuous sections of a fiber bundle A_θ in $M_q(\mathbb{C})$ algebras over a 2-torus

Better way to track the origin of a term (within hundred ones)

The noncommutative torus

In dimension d, with $u = k^2$

Theorem (I.–Masson)

$$
\begin{aligned}
\mathcal{R}_2 = g^{\nu_1\nu_2}\Big(& -(d+2)\,\mathbf{X}_3[u\otimes(\delta_{\nu_1}k)(\delta_{\nu_2}k)\otimes u] + \tfrac{1}{2}(d^2+2d+8)\,\mathbf{X}_4[u\otimes k(\delta_{\nu_1}k)\otimes(\delta_{\nu_2}k)k\otimes u] \\
& -(d-2)\,\mathbf{X}_4[k(\delta_{\nu_1}k)\otimes u\otimes(\delta_{\nu_2}k)k\otimes u] - 2d\,\mathbf{X}_4[k(\delta_{\nu_1}k)\otimes(\delta_{\nu_2}k)k\otimes u\otimes u] \\
& -(d-2)\,\mathbf{X}_4[u\otimes k(\delta_{\nu_1}k)\otimes u\otimes(\delta_{\nu_2}k)k] + 4\,\mathbf{X}_4[k(\delta_{\nu_1}k)\otimes u\otimes u\otimes(\delta_{\nu_2}k)k] \\
& -2d\,\mathbf{X}_4[u\otimes u\otimes k(\delta_{\nu_1}k)\otimes(\delta_{\nu_2}k)k] + \tfrac{1}{2}(d+2)^2\,\mathbf{X}_4[u\otimes k(\delta_{\nu_1}k)\otimes k(\delta_{\nu_2}k)\otimes u] \\
& -(d+2)\,\mathbf{X}_4[k(\delta_{\nu_1}k)\otimes u\otimes k(\delta_{\nu_2}k)\otimes u] - 2(d+2)\,\mathbf{X}_4[k(\delta_{\nu_1}k)\otimes k(\delta_{\nu_2}k)\otimes u\otimes u] \\
& -2(d+2)\,\mathbf{X}_4[u\otimes u\otimes(\delta_{\nu_1}k)k\otimes(\delta_{\nu_2}k)k] - (d+2)\,\mathbf{X}_4[u\otimes(\delta_{\nu_1}k)k\otimes u\otimes(\delta_{\nu_2}k)k] \\
& +\tfrac{1}{2}(d+2)(d+4)\,\mathbf{X}_4[u\otimes(\delta_{\nu_1}k)k\otimes k(\delta_{\nu_2}k)\otimes u]\Big) \\
& +\tfrac{1}{2}d\,\mathbf{X}_3[u\otimes k(\Delta k)\otimes u] - 2\,\mathbf{X}_3[k(\Delta k)\otimes u\otimes u] \\
& -2\,\mathbf{X}_3[u\otimes u\otimes(\Delta k)k] + \tfrac{1}{2}d\,\mathbf{X}_3[u\otimes(\Delta k)k\otimes u]
\end{aligned}
$$

The noncommutative torus

$$
\begin{aligned}
\mathcal{R}_4 = &- 12(d+2)\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, \mathbf{X}_6[u \otimes u \otimes u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u] \\
&+ 2(d^2 + 4d + 8)\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, \mathbf{X}_6[u \otimes u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes u \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u] \\
&+ 4(d^2 + 6d + 16)\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, \mathbf{X}_6[u \otimes u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u \otimes u] \\
&+ 2(d^2 + 4d + 8)\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, \mathbf{X}_6[u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes u \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u \otimes u] \\
&- 12(d+2)\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, \mathbf{X}_6[u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u \otimes u \otimes u] \\
&+ 2(d^2 + 6d + 12)\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, \mathbf{X}_6[u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes u \otimes u \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u] \\
&- 12(d+2)\, g^{\nu_1\nu_4} g^{\nu_2\nu_3}\, \mathbf{X}_6[u \otimes u \otimes u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u] \\
&- 4d\, g^{\nu_1\nu_4} g^{\nu_2\nu_3}\, \mathbf{X}_6[u \otimes u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes u \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u] \\
&+ 4(d^2 + 6d + 16)\, g^{\nu_1\nu_4} g^{\nu_2\nu_3}\, \mathbf{X}_6[u \otimes u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u \otimes u] \\
&- 4d\, g^{\nu_1\nu_4} g^{\nu_2\nu_3}\, \mathbf{X}_6[u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes u \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u \otimes u] \\
&- 12(d+2)\, g^{\nu_1\nu_4} g^{\nu_2\nu_3}\, \mathbf{X}_6[u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u \otimes u \otimes u] \\
&+ 8\, g^{\nu_1\nu_4} g^{\nu_2\nu_3}\, \mathbf{X}_6[u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes u \otimes u \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u] \\
&+ \cdots\cdots
\end{aligned}
$$

$\mathcal{R}_4$ has a presentation with 3527 terms before simplification $\to$ 197 terms