Spectral Triple with Real Structure for Fuzzy Sphere

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The algebra of fuzzy sphere can carry representations of SU(2) Lie group, which is given by M_{2n+1} matrices for its n-th representation. So it is fruitful to study the geometry in context of both matrix models and NCG.
$\rightarrow$Our goal in mind is to take S^2_* to be the finite space and try to formulate gauge theory using the prescription of almost commutative geometry *a la'* Connes.

Result:
First we have tried to find the spectral data for S^2_*, including a real structure, and shown that although the zero-order condition is valid. **But** the *first order condition* is broken with our choice of spectral triple.

Algebra of S^2_*

S^2_* algebra is given by

$$[\hat{x}_i, \hat{x}_j] = i\lambda\epsilon_{ijk}\hat{x}_k; \qquad i, j, k \in \{1, 2, 3\} \tag{1}$$

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$$\mathcal{H} = span_{\mathbb{C}}\Big\{|\frac{1}{2}\rangle, |-\frac{1}{2}\rangle\Big\}$$

where

$$\hat{x}_+|\frac{1}{2}\rangle = \hat{x}_-|-\frac{1}{2}\rangle = 0; \quad \hat{x}_+|-\frac{1}{2}\rangle = \lambda|\frac{1}{2}\rangle; \quad \hat{x}_-|\frac{1}{2}\rangle = \lambda|-\frac{1}{2}\rangle$$
$$\hat{x}_3|\pm\frac{1}{2}\rangle = \pm\frac{\lambda}{2}|\pm\frac{1}{2}\rangle; \quad \hat{x}_\pm = \hat{x}_1 \pm i\hat{x}_2 \tag{2}$$

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Orthonormality and completeness condition for $\mathcal{H}$ -

$$\langle m|n\rangle = \delta_{mn}; \quad |\frac{1}{2}\rangle\langle\frac{1}{2}| + |-\frac{1}{2}\rangle\langle-\frac{1}{2}| := \mathbf{I}_{\mathcal{H}} \;\; ; \quad m,n \in \{\frac{1}{2}, -\frac{1}{2}\} \tag{3}$$

Spectral Triple for S^2_*

Algebra :-

$$\mathcal{A}_F = \mathcal{H} \otimes \tilde{\mathcal{H}} = span_{\mathbb{C}}\{|\frac{1}{2}\rangle\langle\frac{1}{2}|, |\frac{1}{2}\rangle\langle-\frac{1}{2}|, |-\frac{1}{2}\rangle\langle\frac{1}{2}|, |-\frac{1}{2}\rangle\langle-\frac{1}{2}|\} \quad (4)$$

$$= span_{\mathbb{C}}\{\mathbf{1}, \hat{\vec{\sigma}}\} = M_2(C)$$

For example- $|\frac{1}{2}\rangle\langle\frac{1}{2}| = |\frac{1}{2}, \frac{1}{2}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(1 + \sigma_3) \in \mathcal{A}_F$

Note that we can represent $\hat{x}_i$'s in the basis of $\mathcal{H}$ as,

$$\hat{\vec{x}} = \frac{\lambda}{2}\vec{\sigma}$$

Multiplication rule :-

$$\hat{x}_i\hat{x}_j = \frac{\lambda^2}{4}\delta_{ij}\mathbf{1} + i\frac{\lambda}{2}\epsilon_{ijk}\hat{x}_k$$

Spectral Triple for S^2_*

Hilbert space:- Note $\mathcal{H} \otimes \tilde{\mathcal{H}}$, is a bi-module of $\mathcal{A}_F$. The elements $|m\rangle\langle n| := |m, n)$ provide complete orthonormal basis for $\mathcal{A}_F$,

$$(m', n'|m, n) = \delta_{m,m'}\delta_{n,n'}; \quad \sum_{m,n} |m, n)(m, n| = \mathbf{1}_{\mathcal{A}_F} \in \mathcal{A}_F \otimes \tilde{\mathcal{A}}_F \tag{5}$$

So we take our Hilbert space to be

$$\mathcal{H}_F := \mathbb{C}^2 \otimes \mathcal{A}_F = span_{\mathbb{C}}\{|\phi_i)), \quad i = 1, ..., 8\} \tag{6}$$

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Canonical basis:

$$|\phi_1)) = \begin{pmatrix} |\frac{1}{2}\rangle\langle\frac{1}{2}| \\ 0 \end{pmatrix}, \quad |\phi_2)) = \begin{pmatrix} |\frac{1}{2}\rangle\langle-\frac{1}{2}| \\ 0 \end{pmatrix}, \quad |\phi_3)) = \begin{pmatrix} |-\frac{1}{2}\rangle\langle\frac{1}{2}| \\ 0 \end{pmatrix},$$
$$|\phi_4)) = \begin{pmatrix} |-\frac{1}{2}\rangle\langle-\frac{1}{2}| \\ 0 \end{pmatrix}, |\phi_5)) = \begin{pmatrix} 0 \\ |\frac{1}{2}\rangle\langle\frac{1}{2}| \end{pmatrix}, \quad |\phi_6)) = \begin{pmatrix} 0 \\ |\frac{1}{2}\rangle\langle-\frac{1}{2}| \end{pmatrix}$$
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Spectral Triple for S^2_*

Dirac Operator and Chirality Operator:- (U. C Watamura, S Watamura, *Comm. Math. Phys.*, ***212***, *395-413 (2000)*)

$$D_F = \frac{i}{r_n\lambda}\gamma_F \epsilon_{ijk}\sigma_i \hat{x}_j{}^R \hat{x}_k;$$
$$\gamma_F = \frac{1}{\mathcal{N}}(\vec{\sigma}.\hat{\vec{x}}^R - \frac{\lambda}{2}) \to \mathrm{SU(2)\ symmetric} \qquad (9)$$

$$r_n^2 := \hat{x}_i.\hat{x}_i = \frac{3\lambda^2}{4}, \text{ and } \mathcal{N} = \lambda \text{ for } n = \frac{1}{2} \text{ representation}$$

Here $x_j^R\psi = \psi\hat{x}_j$.

$$\{D_F, \gamma_F\} = 0$$

Define- $\hat{J}_i = \hat{L}_i + \frac{\sigma_i}{2}$; $L_i = \frac{1}{\lambda}(\hat{x}_i - \hat{x}_i^R)$

$$[D_F, \hat{J}_i] = 0$$

Eigen-spinors of the Dirac Operator

We can find eigen-spinors of Dirac operators that follow -

$$D_F|m; j, j_3)) = m|m; j, j_3)); \qquad \hat{J}_3|m; j, j_3)) = j_3|m; j, j_3));$$
$$\hat{\vec{J}}^2|m; j, j_3)) = j(j+1)|m; j, j_3)) \tag{10}$$

<u>m=1</u>

$$|\psi_1)) = N \begin{pmatrix} |\frac{1}{2}\rangle\langle\frac{1}{2}| + (\sqrt{3}-1)|-\frac{1}{2}\rangle\langle-\frac{1}{2}| \\ (2-\sqrt{3})|\frac{1}{2}\rangle\langle-\frac{1}{2}| \end{pmatrix} = |1; \frac{1}{2}, \frac{1}{2}\rangle; \; N = \frac{1}{3-\sqrt{3}}$$

$$|\psi_2)) = N \begin{pmatrix} (2-\sqrt{3})|-\frac{1}{2}\rangle\langle\frac{1}{2}| \\ (\sqrt{3}-1)|\frac{1}{2}\rangle\langle\frac{1}{2}| + |-\frac{1}{2}\rangle\langle-\frac{1}{2}| \end{pmatrix} = |1; \frac{1}{2}, -\frac{1}{2}\rangle \tag{11}$$

m=-1

$$|\psi_3)) = N\begin{pmatrix}(1-\sqrt{3})|-\frac{1}{2}\rangle\langle-\frac{1}{2}|+(2-\sqrt{3})|\frac{1}{2}\rangle\langle\frac{1}{2}| \\ |\frac{1}{2}\rangle\langle-\frac{1}{2}|\end{pmatrix} = |-1;\frac{1}{2},\frac{1}{2}\rangle$$

$$|\psi_4)) = N\begin{pmatrix}|-\frac{1}{2}\rangle\langle\frac{1}{2}| \\ (2-\sqrt{3})|-\frac{1}{2}\rangle\langle-\frac{1}{2}|+(1-\sqrt{3})|\frac{1}{2}\rangle\langle\frac{1}{2}|\end{pmatrix} = |-1;\frac{1}{2},-\frac{1}{2}\rangle$$

---

m=0

$$|\psi_5)) = \begin{pmatrix}|\frac{1}{2}\rangle\langle-\frac{1}{2}| \\ 0\end{pmatrix} = |0;\frac{3}{2},\frac{3}{2}\rangle, \qquad |\psi_6)) = \begin{pmatrix}0 \\ |-\frac{1}{2}\rangle\langle\frac{1}{2}|\end{pmatrix} = |0;\frac{3}{2},-\frac{3}{2}\rangle$$

$$|\psi_7)) = \frac{1}{\sqrt{3}}\begin{pmatrix}-|\frac{1}{2}\rangle\langle\frac{1}{2}|+|-\frac{1}{2}\rangle\langle-\frac{1}{2}| \\ |\frac{1}{2}\rangle\langle-\frac{1}{2}|\end{pmatrix} = |0;\frac{3}{2},\frac{1}{2}\rangle,$$
$$|\psi_8)) = \frac{1}{\sqrt{3}}\begin{pmatrix}|-\frac{1}{2}\rangle\langle\frac{1}{2}| \\ |\frac{1}{2}\rangle\langle\frac{1}{2}|-|-\frac{1}{2}\rangle\langle-\frac{1}{2}|\end{pmatrix} = |0;\frac{3}{2},-\frac{1}{2}\rangle \tag{12}$$

These eigen spinors provide a complete orthonormal basis for $\mathcal{H}_F$.

Chiral Basis:-

We can write

$$|\chi^1_\pm)) = \frac{1}{\sqrt{2}}(|\psi_1)) \pm |\psi_3)))$$

$$|\chi^2_\pm)) = \frac{1}{\sqrt{2}}(|\psi_2)) \pm |\psi_4)))$$

It can be shown,

$$\begin{aligned} \gamma_F|\chi^i_+)) &= |\chi^i_+)), \quad i = 1,2 \\ \gamma_F|\chi^i_-)) &= -|\chi^i_-)), \quad i = 1,2 \\ \gamma_F|\psi_i)) &= -|\psi_i)), \quad i = 5,6,7,8 \end{aligned} \tag{13}$$

$\mathcal{H}_F = C^2 \otimes \mathcal{H} \otimes \tilde{\mathcal{H}}$, which is spanned by the Dirac spinors (alternatively the chiral spinors) can be shown to split, through Clebsch-Gordon rule, into a quadruplate and a pair of doublet states as- $2 \times (2 \times 2) = 4 \oplus 2 \oplus 2$.

Determination of $\mathcal{J}_F$

$\mathcal{J}_F$ (anti-unitary isomorphism) should be such that it satisfies the properties of a particular KO dimension-

n	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

where, $\mathcal{J}_F^2 = \epsilon$; $\mathcal{J}_F D_F = \epsilon' D_F \mathcal{J}_F$; $\mathcal{J}_F \gamma_F = \epsilon'' \gamma_F \mathcal{J}_F$; $\epsilon, \epsilon', \epsilon'' \in \{1, -1\}$
Also,

$$\mathcal{J}_F \pi(a)^* \mathcal{J}_F^* = \pi(a)^o \;\; \forall \; a \; \in \; \mathcal{A}_F$$

so that $[\pi(a), \pi(b)^o] = 0, \;\; \forall \; a, b \; \in \; \mathcal{A}_F$ (zero order condition)

$[[D, \pi(a)], \pi(b)^o] = 0$ (1st order condition)

Determination of $\mathcal{J}_F$

Existence of γ_F $\rightarrow$ even KO dimension $\rightarrow$ $\epsilon' = 1$ $\rightarrow$ $[D_F, \mathcal{J}_F] = 0$

$$\mathcal{J}_F : |\pm 1, \frac{1}{2}, \frac{1}{2})) \quad \longleftrightarrow \quad \pm|\pm 1, \frac{1}{2}, -\frac{1}{2})) \tag{14}$$

$$I_D = dim(H_+) - dim(H_-) \;= -4 \neq 0$$

J.Barrett , *J. Math. Phys.* ***56***, *082301 (2015)*

$$[\mathcal{J}_F, \gamma_F] = 0 \;\text{ for KO dim -0,4} \tag{15}$$

$$\boxed{\mathcal{J}_F\psi_1 = \psi_2; \quad \mathcal{J}_F\psi_2 = \pm\psi_1; \quad \mathcal{J}_F\psi_3 = \psi_4; \quad \mathcal{J}_F\psi_4 = \pm\psi_3}$$

As a sample let us write,

$$|\psi_1)) = N \begin{pmatrix} |\frac{1}{2}\rangle\langle\frac{1}{2}| + (\sqrt{3}-1)|-\frac{1}{2}\rangle\langle-\frac{1}{2}| \\ (2-\sqrt{3})|\frac{1}{2}\rangle\langle-\frac{1}{2}| \end{pmatrix} = |1; \frac{1}{2}, \frac{1}{2}\rangle$$

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Action of $\mathcal{J}_F$-sector-wise in $\mathcal{A}_F = L_3^{\pm 1} \oplus L_3^0$:

Swaps the $L_3 = \pm 1$ states in $\mathcal{A}_F$ as,

$$|\frac{1}{2}\rangle\langle-\frac{1}{2}| \longleftrightarrow (\pm)|-\frac{1}{2}\rangle\langle\frac{1}{2}| \qquad (17)$$

Swaps between $L_3 = 0$ states in $\mathcal{A}_F$ as ,

$$|\frac{1}{2}\rangle\langle\frac{1}{2}| \longleftrightarrow (\pm)|-\frac{1}{2}\rangle\langle-\frac{1}{2}| \qquad (18)$$

Simultaneously exchange in upper and lower components indicates that $\mathcal{J}_F$ acts non-trivially on the C^2 sector.

Determination of $\mathcal{J}_F$

In the massless sector-

$$\mathcal{J}_F\psi_5 = \psi_6; \;\; \mathcal{J}_F\psi_6 = \pm\psi_5; \;\; \mathcal{J}_F\psi_7 = \psi_8; \;\; \mathcal{J}_F\psi_8 = \pm\psi_7$$

These suggest that we should define $\mathcal{J}_F$ as following:

$$\mathcal{J}_F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes (H \circ P) \tag{19}$$

where H is the Hermitian conjugation operation and P is the space reversal transformation:

$$P : \hat{x}_i \to -\hat{x}_i \tag{20}$$

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Under P,however,the fuzzy sphere algebra $\to [\hat{x}_i, \hat{x}_j] = i\lambda\epsilon_{ijk}\hat{x}_k$ is **non-invariant** .

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We elevate the Levi-civita(which is a pseudo tensor) to the level of a tensor, which is defined as

$$\begin{aligned} E_{ijk} &= \epsilon_{ijk}, \quad \text{for right handed system} \\ &= -\epsilon_{ijk} \quad \text{for left handed system} \end{aligned}$$

Transformation rule under parity as: $\boxed{P : E_{ijk} \to E'_{ijk} = -E_{ijk}}$

$$[\hat{x}_i, \hat{x}_j] = i\lambda E_{ijk}\hat{x}_k$$

remains invariant under P. $\therefore$ Total symmetry group is $O(3)$

G. Fiore and F. Pisacane, DOI: https://doi.org/10.22323/1.318.0184

Modified multiplication rule -

$$\hat{x}_i\hat{x}_j = \frac{\lambda^2}{4}\delta_{ij}\mathbf{1} + i\frac{\lambda}{2}E_{ijk}\hat{x}_k$$

With this

$$\begin{aligned} &\mathcal{J}_F\psi_2 = \psi_1; \;\; \mathcal{J}_F\psi_1 = -\psi_2 \;\; \mathcal{J}_F\psi_3 = -\psi_4; \;\; \mathcal{J}_F\psi_4 = \psi_3; \\ &\mathcal{J}_F\psi_5 = \psi_6; \;\; \mathcal{J}_F\psi_6 = -\psi_5; \;\; \mathcal{J}_F\psi_7 = -\psi_8; \;\; \mathcal{J}_F\psi_8 = \psi_7; \end{aligned} \tag{21}$$

This implies $\mathcal{J}_F^2 = -1 \quad \to$ KO dimension- 4. Using this we can show, for example

$$(H\circ P)\Big(\hat{x}_i\Big|\frac{1}{2}\Big\rangle\Big\langle\frac{1}{2}\Big|\Big) = (H\circ P)(\hat{x}_i(\frac{1}{2} + \frac{\hat{x}_3}{\lambda})) = (H\circ P)(\frac{\hat{x}_i}{2} + \frac{\hat{x}_i\hat{x}_3}{\lambda}) = H(-\frac{\hat{x}_i}{2} + \frac{\hat{x}_i\hat{x}_3}{\lambda})$$

$$= -\frac{\hat{x}_i}{2} + \frac{\hat{x}_3\hat{x}_i}{\lambda} = -\hat{x}_i^R(\frac{1}{2} - \frac{\hat{x}_3}{\lambda}) = -\hat{x}_i^R\Big| -\frac{1}{2}\Big\rangle\Big\langle -\frac{1}{2}\Big|$$

Using this ,

$$\mathcal{J}_F \pi(\hat{x}_1)\mathcal{J}_F^\dagger \psi_1 = \mathcal{J}_F \pi(\hat{x}_1)\psi_2; \qquad \pi(\hat{\vec{x}}) = \begin{pmatrix} \hat{\vec{x}} & 0 \\ 0 & \hat{\vec{x}} \end{pmatrix}$$

$$= \mathcal{J}_F N \begin{pmatrix} (2-\sqrt{3})\frac{\hat{x}_1\hat{x}_-}{\lambda} \\ (\sqrt{3}-1)(\frac{\hat{x}_1}{2}+\frac{\hat{x}_1\hat{x}_3}{\lambda}) + (\frac{\hat{x}_1}{2}-\frac{\hat{x}_1\hat{x}_3}{\lambda}) \end{pmatrix}$$

$$= N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\hat{x}_+\hat{x}_1}{\lambda} \\ (\sqrt{3}-1)(-\frac{\hat{x}_1}{2}+\frac{\hat{x}_3\hat{x}_1}{\lambda}) - (\frac{\hat{x}_1}{2}+\frac{\hat{x}_3\hat{x}_1}{\lambda}) \end{pmatrix} = -\pi(\hat{x}_1^R)\psi_1$$

For all generators of algebra,

$$\pi(\hat{x}_i^o) = \mathcal{J}_F \pi(x_i)\mathcal{J}_F^\dagger = -\pi(\hat{x}_i^R)$$

This helps us to identify: $\boxed{\hat{x}_i^o = -\hat{x}_i^R = P(\hat{x}_i^R) \quad \text{and not} \quad \hat{x}_i^o = \hat{x}_i^R}$

Finally

$$[\hat{x}_i^o, \hat{x}_j^o] = -i\lambda E_{ijk}\hat{x}_k^o \quad \rightarrow \quad SU(2)^R \text{ algebra}$$

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- $L_i = \frac{1}{\lambda}(\hat{x}_i - \hat{x}_i^R) \xrightarrow{P} -L_i$ (under P operation).
  In our definition the commutative counterpart of $L_i = E_{ijk}\hat{x}_j\hat{p}_k$, which also changes sign under parity.
  $\hat{L}$ is a vector rather a pseudovector in our formulation.

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  $\hat{L}$ is a vector rather a pseudovector in our formulation.
- $\hat{J}_i = L_i + \frac{\sigma_i}{2}$ should also transform the same way, under parity as L_i does. So, we have to impose $\sigma_i \xrightarrow{P} -\sigma_i$.

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  $\hat{L}$ is a vector rather a pseudovector in our formulation.
- $\hat{J}_i = L_i + \frac{\sigma_i}{2}$ should also transform the same way, under parity as L_i does. So, we have to impose $\sigma_i \xrightarrow{P} -\sigma_i$.
- **Chirality operator:** $\gamma_F = \frac{1}{\lambda}(\sigma_i\hat{x}_i^R - \frac{\lambda}{2}) \xrightarrow{P} \gamma_F$

Finally

$$[\hat{x}_i^o, \hat{x}_j^o] = -i\lambda E_{ijk}\hat{x}_k^o \quad \rightarrow \quad SU(2)^R \text{ algebra}$$

Some key points:

- $L_i = \frac{1}{\lambda}(\hat{x}_i - \hat{x}_i^R) \xrightarrow{P} -L_i$ (under P operation).
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- **Chirality operator:** $\gamma_F = \frac{1}{\lambda}(\sigma_i\hat{x}_i^R - \frac{\lambda}{2}) \xrightarrow{P} \gamma_F$
- **Dirac Operator:**

$$D_F = \frac{i}{r_n\lambda}\gamma_F E_{ijk}\sigma_i\hat{x}_j{}^R\hat{x}_k \quad \xrightarrow{P} \quad \frac{i}{r_n\lambda}\gamma_F(-E_{ijk})(-\sigma_i)(-\hat{x}_j{}^R)(-\hat{x}_k) = D_F$$

First Order Condition

$$[[D, \pi(a)], \pi(b)^o] =^? 0$$

Without loss of generality, we can take
$a = \hat{x}_l, \pi(\hat{x}_l) = diag(\hat{x}_l, \hat{x}_l)$ and $b^o = -\hat{x}_p^R, \pi(\hat{x}_p^o) = diag(-\hat{x}_p^R, -\hat{x}_p^R)$.

$$[[D, \pi(\hat{x}_l)], \pi(\hat{x}_p)^o] =$$

$$= \frac{i\lambda\gamma_F}{r_n}\left(\epsilon_{lpm}\sigma_i\hat{x}_i\hat{x}_m^R - \epsilon_{jpm}\sigma_l\hat{x}_j\hat{x}_m^R\right) \neq 0$$

The first order condition is violated.

Future prospect

Since the above mentioned spectral triple violates the first order condition - an important ingredient in spectral formulation of standard model, this opens the door to investigate phenomena beyond standard model through a toy model.

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$\rightarrow$ In the formulation of Standard model with almost commutative geometry, a strong restriction is implemented through the **First Order Condition** requiring that the Dirac operator is a differential operator of order one.

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Since the above mentioned spectral triple violates the first order condition - an important ingredient in spectral formulation of standard model, this opens the door to investigate phenomena beyond standard model through a toy model.

$\rightarrow$ In the formulation of Standard model with almost commutative geometry, a strong restriction is implemented through the **First Order Condition** requiring that the Dirac operator is a differential operator of order one.

$\rightarrow$ Without this restriction, the invariance of the fluctuated Dirac operator, under inner-automorphism of the algebra, is violated.
To restore the invariance one has to add another term to the fluctuated Dirac operator $\Rightarrow$ **Beyond SM physics**.
(A. H. Chamseddine, A. Connes, W D van Suijlekom, J. Geo. Phys., **73**(2013)222-234; JHEP 1311 (2013) 132)

Conclusion

- We have tried to provide a consistent formulation of an even and real spectral triple for the fuzzy sphere in its $1/2$ representation using Watamura's prescription of Dirac and grading operator.
- We started by obtaining the explicit expressions of the eigen spinors of SU(2) covariant forms of Dirac and chirality operators.
- Finally, we demonstrate how the real structure operator in spin-$1/2$ representation, consistent with the spectral data of KO dimension-4 can be obtained.
- Symmetry group is now enhanced to O(3) from SO(3).
- The first order condition is violated.

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THANK YOU