NONCOMMUTATIVE U(1) DEFORMATIONS AND L_∞ BOOTSTRAP

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OUTLINE

- NC GAUGE THEORY (ONE STANDARD APPROACH : DUBOIS-VIOLETTE - MICHOR)
  NC-U(1) case and Moyal example

- NOVEL APPROACH BASED ON SYMPLECTIC EMBEDDINGS (Szabo & Kupriyanov)

  - Deformed gauge transformations
  - Deformed field strength
  - Examples with Lie-algebra type NC
  - Connection with homotopy algebras and L_∞ bootstrap

- SUMMARY, PERSPECTIVES .....

WHAT IS THE PROBLEM WE WANT TO ADDRESS?

Motivations for NC field theory date back to many decades ago, but still valid

There is no general consensus about the more appropriate formulation, and at some stage we face problems

- Consistent formulation, compatible with commutative limit.
- appropriate differential calculus
- UV / IR mixing

"CLASSICAL" NC GAUGE THEORY

- $(\mathcal{A}, \star)$ NC algebra representing space-time
- $\mathbb{M}$ right $\mathcal{A}$-module representing matter fields (replaces vector bundles)
- Unitary automorphisms of $\mathbb{M}$ representing gauge transformations
- Dynamics described by means of derivations based differential calculus

Gauge connection $\quad \nabla : \mathrm{Der}(\mathcal{A}) \times \mathbb{M} \to \mathbb{M}$

$$\nabla_X (mf) = \nabla_X m\, f + m\, X(f)$$

s.t.

$$X\left(h(m_1, m_2)\right) = h(\nabla_X m_1, m_2) + h(m_1, \nabla_X m_2)$$

with h Hermitean structure

$$h : \mathbb{M} \times \mathbb{M} \to \mathcal{A}$$

Curvature : $F(X,Y) : \mathbb{M} \longrightarrow \mathbb{M}$

$$F(X,Y)\,m = [\nabla_X, \nabla_Y]\,m - \nabla_{[X,Y]}\,m \qquad X,Y \in \mathrm{Der}(\mathcal{A})$$

Gauge transformations:

automorphisms of $\mathbb{M}$ compatible with h

$g \in \mathcal{U}(\mathbb{M})$ $\qquad g(mf) = g(m)\,f$

$m_1, m_2 \in \mathbb{M}$; $f \in \mathcal{A}$ $\qquad h\,(g(m_1), g(m_2)) = h\,(m_1, m_2)$

such that $\qquad \nabla^g_X = g^{-1} \circ \nabla_X \circ g$

$$F^g(X,Y) = g^{-1} \circ F(X,Y) \circ g$$

For NC $U(1)$:

commutative matter fields are sections of complex line bundle $\Longrightarrow$ NC generalization is 1-dimensional $\mathcal{A}$-module

$$\mathbb{M} = \mathbb{C} \otimes \mathcal{A}$$

$m \in \mathbb{M}$ identified with $\mathbb{1} f$

$$h(m_1, m_2) = h(f_1, f_2) = f_1^+ f_2$$

∇_X determined by action on $\mathbb{1}$ $\quad \nabla_X(\mathbb{1} f) = \nabla_X(\mathbb{1}) f + X(f)$

connection one form $\quad A : X \to A(X) := i \nabla_X(\mathbb{1})$

gauge transformations: $\quad g(\mathbb{1} f) = g(\mathbb{1}) \star f$

$$h(g(\mathbb{1} f_1), g(\mathbb{1} f_2)) = f_1^+ f_2$$

implies $\quad g(\mathbb{1})^+ \star g(\mathbb{1}) = 1$

pose $\quad g(\mathbb{1}) = g \in \mathcal{U}(\mathcal{A})$ unitary elems of $\mathcal{A}$ acting multiplicatively from the left

Ex: $(\mathcal{A}, \star) = \mathbb{R}^2_\theta$ Moyal plane

- $\mathrm{Der}(\mathcal{A}) \longrightarrow$ generated by $\dfrac{\partial}{\partial x^\mu}$

- $F_{\mu\nu} = F(\partial_\mu, \partial_\nu) = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star \rightarrow (1)$

$(A_\mu = i \nabla_\mu(\mathbb{1}))$

- gauge transformations: $\mathcal{U}(\mathbb{R}^2_\theta) \ni g = \exp_\star(if)$

$$A_\mu^g = g \star A_\mu \star g^\dagger - i \partial_\mu g \star g^\dagger$$

$$F_{\mu\nu}^g = g \star F_{\mu\nu} \star g^\dagger$$

$g = \exp_\star(if)$ unitaries of $\mathbb{R}^2_\theta \rightarrow$

$$\delta_f A_\mu = \partial_\mu f + i[f, A_\mu]_\star \qquad \delta_f F_{\mu\nu} = i[f, F_{\mu\nu}]_\star$$

(1) $F(X,Y)m = \nabla_X \nabla_Y m - \nabla_Y \nabla_X m - \nabla_{[X,Y]} m$

$X = \partial_\mu, \quad Y = \partial_\nu \qquad [X,Y] = 0 \qquad ; \qquad m = \mathbb{1} f$

$$\nabla_X \nabla_Y m = \nabla_X \left(\nabla_Y (\mathbb{1}) f + \mathbb{1} Y(f) \right) = \nabla_X \left(\mathbb{1} A(Y) f + \mathbb{1} Y(f) \right)$$

$$= \nabla_X (\mathbb{1}) A(Y) f + \mathbb{1} X\left(A(Y) f\right) + \nabla_X (\mathbb{1}) Y(f) + \mathbb{1} X(Y(f))$$

$$= \mathbb{1} \Big\{ A(X) A(Y) f + X(A(Y)) f + \cancel{A(Y) X(f)} + \cancel{A(X) Y(f)} + X(Y(f)) \Big\}$$

$$\nabla_Y \nabla_X m = \mathbb{1} \Big\{ A(Y) A(X) f + Y(A(X)) f + \cancel{A(X) Y(f)} + \cancel{A(Y) X(f)} + Y(X(f)) \Big\}$$

$$\Rightarrow \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X \right) m = \mathbb{1} \Big\{ \left[A(X) A(Y) - A(Y) A(X) \right] + X(A(Y)) - Y(A(X)) \Big\} f$$

The generalisation to $U(N)$ requires $\underline{M} = \mathbb{C}^N \otimes \mathcal{A}$

The first two questions addressed $\left\{ \begin{array}{ll} \text{commutative limit} & a) \\ \text{differential calculus} & b) \end{array} \right.$

are fulfilled by Moyal NC space-time

In general do not work properly for non-constant non-commutativity

Already for $[x^\mu, x^\nu]_\star = \Theta^{\mu\nu}(x)$ with $\Theta^{\mu\nu}(x)$ linear in x

the commutative limit is not correct ($\Rightarrow$ ~~a)~~)

the derivation based differential calculus not adequate to describe dynamics ~~b)~~

Alternatives : modify the differential calculus (e.g. twist)

modify the action functional

⋮

Approach presented in this talk:

- Work in semi-classical approx. [star-commutator replaced by Poisson bracket : $(\mathcal{A}, \star_\Theta) \longrightarrow (\mathcal{F}(M), \{\ ,\ \}_\Theta)$]
- Reinterpret standard (commutative) gauge theory within symplectic geometry of (T^*M, ω_0) ("canonical" symplectic embedding)
- Introduce new ω s.t. $\pi_*\, \omega^{-1} = \Theta$ $\quad \Theta = \Theta^{\mu\nu}\, \partial_\mu \wedge \partial_\nu$ so to embed non trivial PB on M, $\{\ ,\ \}_\Theta$ (symplectic embedding of deformed PB)
- Define gauge transformations, field strength and dynamics, within the modified symplectic embedding

• Standard $U(1)$ gauge theory revisited

$(T^*M\ ,\ \omega_0)$ $\qquad \omega_0 = d\lambda_0 \quad \lambda_0$ Liouville one-form

gauge potential $\quad A \in \Omega^1(U) \qquad U \subset M$

A is associated with a local section

$S_A : U \to T^*U$ through the local trivialisation

$$\psi_U : U \to \pi^{-1}(U)$$

$$\psi_U^{-1}(S_A(x)) = (x, A(x)) \qquad x \in U$$

The image of S_A is a submanifold of T^*U

Consider

$$\xi_A = \lambda_0 - \pi^* A$$

local one-form on T^*U

By definition λ_0 s.t. $\boxed{s_A^*(\lambda_0) = A}$

(given $s_i : x \to u \in T^*M$ s.t. $\psi_i^{-1}(s_i(x)) = (x, \alpha(x))$

λ_0 is the tautological one-form namely $s_i^*(\lambda_0)(x) = \alpha(x)$

but also $s_A^*(\pi^* A) = (\pi \circ s_A)^* A = A$

$\Rightarrow$ $s_A^*(\xi_A) = 0$ $\quad$ ξ_A vanishes exactly on $\mathrm{im}(s_A) \subset T^*U$

in local coordinates $\xi_A \to p_\mu dx^\mu - A_\mu dx^\mu$

$s_A^*(\xi_A) = 0 \longrightarrow$ fixes the fibre coordinate of $(x, p(x))$ at $p = A$

$\Rightarrow$ The infinitesimal gauge transformation of A with gauge parameter f may be defined in terms of ω_0^{-1}

$$\delta_f A_\nu(x) := s_A^* \{ \pi^* f, \xi_{A_\nu} \}_{\omega_0^{-1}} = \frac{\partial f}{\partial x^\rho} \frac{\partial (p_\nu - A_\nu)}{\partial p_\rho}$$

$$= \frac{\partial}{\partial x^\nu} f$$

To summarize : we may say that, in order to compute $\delta_f A_\nu(x)$

- we have embedded the trivial PB $\{x^\mu, x^\nu\} = 0$ on M into the canonical one on T^*M
- we have identified A with the fibre coordinate of a local section on T^*M
- we have computed PB's on T^*M
- and pulled-back to M

We can add :

$$\delta_f \delta_g A - \delta_g \delta_f A = 0$$

namely

$$\boxed{[\delta_f, \delta_g] A = \delta_{\{f,g\}_0} A}$$

it being $\{f(x), g(x)\}_0 = 0$

- The algebra of gauge transformations closes with respect to the (trivial) Poisson bracket on M

- The conjugate momentum, p, is an auxiliary variable, disappeared from the gauge algebra.

$\Longrightarrow$ Generalization to non-trivial PB on M

Symplectic realizations (SR), Symplectic embeddings (SE)

SR (Weinstein '83) (T^*M, ω_0) cotangent bundle

ω_0 canonical symplectic form, locally $\omega_0 = d\lambda_0 = dp_\mu \wedge dx^\mu$

λ_0 Liouville one-form

$$\Theta = \Theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu}$$

non-trivial PB on M representing our semiclassical approx of NC space-time

One defines: $X^\Theta \in \mathfrak{X}(M)$

$$X^\Theta = \Theta(\lambda_0, \cdot) = \Theta^{\mu\nu}(x)\, p_\mu \frac{\partial}{\partial x^\nu}$$

associated flow: φ_t^Θ, $t \in \mathbb{R}$

In terms of φ_t^{Θ} it is possible to construct a new symplectic form on M, at least locally:

$$\omega = \int_0^1 (\varphi_t^{\Theta})^*(\omega_0)\, dt$$

Weinstein '83

Crainic & Marcut '11

such that $\pi_*(\omega^{-1}) = \Theta$

it projects down to Θ, PB on M.

In local coordinates $\omega = dp_\mu \wedge dy^\mu$

with $y^\mu = \int_0^1 x^\mu \circ \varphi_t^{\Theta}\, dt$

the Jacobian $(J^\mu{}_\nu) = \left(\frac{\partial y^\mu}{\partial x^\nu}\right)$ is formally invertible

denote $\gamma(x,p) = J^{-1}$

$$\Rightarrow \quad \omega^{-1} = \Theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} + \gamma^\mu{}_\nu(x,p) \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial p_\nu}$$

Generalization (Symplectic Embedding)

Define Λ as a deformation of Θ

$$\Lambda = \Theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} + \gamma^\mu{}_\nu(x,p) \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial p_\nu}$$

Satisfying Jacobi identity $\quad [\Lambda, \Lambda]_S = 0$

provided Θ does ($[\Theta, \Theta]_S = 0$)

This produces an equation for the unknown γ:

$$\gamma^\mu{}_\nu \frac{\partial}{\partial p_\nu} \gamma^\xi{}_\lambda - \gamma^\xi{}_\nu \frac{\partial}{\partial p_\nu} \gamma^\mu{}_\lambda + \Theta^{\mu\nu} \frac{\partial}{\partial x^\nu} \gamma^\xi{}_\lambda - \Theta^{\xi\nu} \frac{\partial}{\partial x^\nu} \gamma^\mu{}_\lambda - \gamma^\nu{}_\lambda \frac{\partial}{\partial x^\nu} \Theta^{\mu\xi} = 0$$

Provided we find solutions for the matrix γ, gauge transformations are defined as in the canonical case:

$$\delta_f A_\nu(x) = s_A^* \{\pi^* f, \xi_{A_\nu}\}_\Lambda$$

with $\xi_A = \lambda_0 - \pi^* A \xrightarrow[\text{locally}]{} p_\mu dx^\mu - A_\mu dx^\mu$

and Λ the non trivial PB with $\pi_* \Lambda = \Theta$

this yields:

$$\delta_f A_\nu(x) = -\{f, A_\nu\}_\Theta + \gamma^\mu_\rho(A) \frac{\partial f}{\partial x^\mu} \frac{\partial}{\partial p_\rho} p_\nu = \gamma^\mu_\nu(A) \frac{\partial f}{\partial x^\mu} - \{f, A_\nu\}_\Theta$$

"Commutative" limit:

$$\left.\begin{array}{l} \Theta \to 0 \\ \gamma^\mu{}_\nu \to \delta^\mu{}_\nu \end{array}\right\} \Rightarrow \delta_f A_\nu(x) = \frac{\partial f}{\partial x^\nu}$$

Closure of the gauge algebra:

$$[\delta_f, \delta_g] A_\nu(x) = -\{\{f, g\}_\Theta, A_\nu\}_\Theta + \gamma^\mu{}_\nu \frac{\partial}{\partial x^\nu} \{f, g\}_\Theta$$

namely : $[\delta_f, \delta_g] A = \delta_{\{f,g\}_\theta} A$

it also satisfies $\delta_{f\cdot g} A_\mu = f \delta_g A + g \delta_f A$

as in the commutative case, $\gamma^\mu_\nu(p) \to \gamma^\mu_\nu(A)$

namely the auxiliary variable p has disappeared.

Notice that for Moyal NC $\Theta = \text{const} \Longrightarrow \partial_\mu \Theta^{\rho\sigma} = 0$

$\Rightarrow \gamma^\mu_\nu(A) = \delta^\mu_\nu$

yielding back

$$\delta_f A_\nu = \partial_\nu f - \{f, A_\nu\}_\Theta$$

($\gamma^\mu_\nu = \delta^\mu_\nu$ also solution of the fully NC case $\to$

$$\delta_f A_\nu = \partial_\nu f - [f, A_\nu]_\star$$
)

Solutions for γ^μ_ν have been found for linear, Lie algebra type non-commutativity. Details later (If there is time)

What about the field strength?

Let us return to the classical case $\Theta = 0$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ may be rewritten in terms of canonical PB as

$$F_{\mu\nu} = S_A^* \{ \xi_{A_\mu}, \xi_{A_\nu} \}_{\omega_0'} = -\frac{\partial(p_\mu - A_\mu)}{\partial x^\rho} \frac{\partial(p_\nu - A_\nu)}{\partial p_\rho} + \frac{\partial(p_\nu - A_\nu)}{\partial x^\rho} \frac{\partial(p_\mu - A_\mu)}{\partial p_\rho}$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu$$

$\Rightarrow$ A natural generalization when $\{x^\mu, x^\nu\} \neq 0$ would be to use the symplectic embedding again:

$$F_{\mu\nu} = S_A^* \{ \xi_{A_\mu}, \xi_{A_\nu} \}_\Lambda = \{ A_\mu(x), A_\nu(x) \}_\Theta$$

$$- \gamma_\sigma^\rho(A) \frac{\partial A_\mu}{\partial x^\rho} \frac{\partial p_\nu}{\partial p_\sigma} + \gamma_\sigma^\rho(A) \frac{\partial A_\nu}{\partial x^\rho} \frac{\partial p_\mu}{\partial p_\sigma}$$

namely :

$$\tilde{\mathcal{F}}_{\mu\nu}(x) = \{ A_\mu(x) , A_\nu(x) \}_\Theta + \gamma^\rho_\mu \, \partial_\rho A_\nu - \gamma^\rho_\nu \, \partial_\rho A_\mu$$

and compute its gauge transformation according to the definition

$$\delta_f \tilde{\mathcal{F}}_{\mu\nu}(x) = s^*_A \{ \pi^* f , \pi^* \tilde{\mathcal{F}}_{\mu\nu} \}_\Lambda$$

However, this definition is not covariant :

$$\delta_f \tilde{\mathcal{F}}_{\mu\nu}(x) = - \{ f , \tilde{\mathcal{F}}_{\mu\nu}(x) \}_\Theta + \text{unwanted terms}$$

$\Rightarrow$ One has to modify the definition of $\mathcal{F}$.

It turns out that

$$\mathcal{F}_{\mu\nu} = R_{\mu\nu}^{\alpha\beta}(A, x) \, \tilde{\mathcal{F}}_{\alpha\beta}$$

transforms covariantly if :

If the unknown function $R^{\alpha\beta}_{\mu\nu}$ satisfies the second master equation:

$$\gamma^{\xi}_{\sigma}\frac{\partial}{\partial A_\sigma}R^{\rho\omega}_{\mu\nu}+\Theta^{\xi\sigma}\partial_\sigma R^{\rho\omega}_{\mu\nu}+R^{\rho\sigma}_{\mu\nu}\frac{\partial}{\partial A_\omega}\gamma^{\xi}_{\sigma}+R^{\sigma\omega}_{\mu\nu}\frac{\partial}{\partial A_\rho}\gamma^{\xi}_{\sigma}=0$$

$R^{\rho\omega}_{\mu\nu} \longrightarrow \delta^{\rho}_{\mu}\delta^{\omega}_{\nu}-\delta^{\omega}_{\mu}\delta^{\rho}_{\nu}$ in the commutative limit.

What is the meaning of this deformation through $R^{\alpha\beta}_{\mu\nu}$ within the symplectic embedding?

Unclear to me

(see Kupriyanov & Szabo '21)

Anyways, with γ and R satisfying the two master eq's I have shown $\Rightarrow$

- F fully covariant
- right commutative limit
- the Poisson algebra of gauge transformations closes

2nd and 3rd point fit nicely into symplectic embedding.

1st to be better understood.

It is possible to define a Lagrangian, which turns out to be covariant and with right commutative limit at least for linear PB's.

$$\mathcal{L} := -\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} \eta^{\mu\rho} \eta^{\nu\sigma}$$

$$\delta_f \mathcal{L} := -\frac{1}{4} \lim_{\varepsilon \to 0} \left(F_{\mu\nu} + \varepsilon \delta_f F_{\mu\nu} \right) \left(F_{\rho\sigma} + \varepsilon \delta_f F_{\rho\sigma} \right) \eta^{\mu\nu} \eta^{\nu\sigma}$$

$$= \dots = -\{ f, \mathcal{L} \}_\Theta$$

and this is a total divergence at least for linear NC

$$\Theta^{\mu\nu} = \theta f^{\mu\nu}{}_\rho \, x^\rho$$

$\Rightarrow$ We have an invariant action with correct commutative limit provided we find solutions for γ^μ_ν and $R^{\alpha\beta}_{\mu\nu}$

Explicit solutions found for 4-dim cases with commuting time and $[x^i, x^j]_\star = f^{ij}{}_k \, x^k$

Connection with L_∞ bootstrap approach

L_∞ algebras are homotopy generalisations of Lie algebras defined on graded vector space

$$V = \bigoplus V_k$$

$k \in \mathbb{Z}$ is the grading of V_k

$$l_n : (v_1, \dots, v_n) \in V^{\otimes n} \longrightarrow v \in V$$

graded antisymmetric
and satisfy generalised Jacobi identities.

For $V = V_0 \oplus V_{-1}$ $f \in V_0$ gauge parameters
$A \in V_{-1}$ gauge potentials

set $l_1(f) = df$ (classical data)
$l_2(f, g) = \{f, g\}$ (deformation)

$\Longrightarrow$ by looking for closure of the algebra one builds $\delta_f A$ which is searched in the form

$$\delta_f A = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-2)}{2}} \ell_{n+1}(f, A, \ldots, A) \qquad (1)$$

The completion of the algebra starting from $\ell_1(f)$, $\ell_2(f,g)$ is the L_∞ bootstrap.

It is proven (Kupriyanov & Szabo) that there exists a one to one correspondence between L_∞-bootstrap and symplectic embeddings.

The correspondence is explicitly given by expanding our previous

$$\delta_f A = -\{f, A_\mu\}_\Theta \, dx^\mu + \gamma^\rho_\mu(A)\, \partial_\rho f \, dx^\mu \qquad \text{in powers of } A$$

and comparing with (1)

Summary

- I have illustrated a constructive procedure to build semiclassical approximations of NC $U(1)$ gauge theory through symplectic embedding.

  correct commutative limit ✓
  covariant field strength and YM Lagrangian ✓
  closure of Poisson gauge algebra ✓
  explicit solutions for 4-dim linear NC ✓
  connection to L_∞ bootstrap ✓

- Geometric meaning of $\mathcal{F}$?
- Extension to a full gauge theory with matter fields (Kupriyanov & Szabo)
- Non-uniqueness of solutions

⋮