Topological Insulators at Strong Disorder

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Part 1

Spectacular behavior at strong disorder

- An exactly solvable system
- Numerical simulations
- Experimental signature of such phenomena

Anderson Localization-Delocalization transition in 1D chiral model

The model defined:

Data:

- Ergodic dynamical system $\left(\tau : \mathbb{Z} \to \mathrm{Homeo}(\Omega), \mathrm{d}\mathbb{P}\right)$
- Two functions $t : \Omega \to \mathbb{R}$ and $m : \Omega \to \mathbb{R}$.

From this data, we assemble a disordered Hamiltonian on $\mathbb{C}^2 \otimes \ell^2(\mathbb{Z})$:

$$H_\omega = \sum_{x \in \mathbb{Z}} \left\{ \frac{1}{2} t_x \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes |x\rangle\langle x+1| + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes |x+1\rangle\langle x| \right] + m_x \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \otimes |x\rangle\langle x|. \right\}$$

with

$$t_x = t(\tau_x \omega), \quad m_x = m(\tau_x \omega)$$

Key symmetry:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} H_\omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -H_\omega.$$

Task: We are going to solve $H\psi = E\psi$ at $E = 0$.

Lyapunov/Anderson localization length

The Schroedinger equation at $E = 0$ reduces to ($\alpha = \pm 1$ indexes the top/bottom of ψ)

$$t_x \psi^{\alpha}_{x-\alpha} + i\alpha m_x \psi^{\alpha}_x = 0 \;\Rightarrow\; \psi^{\alpha}_x = \prod_{j=1}^{x} \left(\frac{t_x}{m_x} \right) \psi^{\alpha}_0 .$$

The Lyapunov exponent (= inverse of Anderson localization length) comes to be

$$\Lambda^{-1} = \max_{\alpha=\pm} \left[- \lim_{x\to\infty} \tfrac{1}{x} \log |\psi^{\alpha}_x| \right] = \left| \lim_{x\to\infty} \tfrac{1}{x} \sum_{n=1}^{x} \Big(\ln |t(\tau_x \omega)| - \ln |m(\tau_x \omega)| \right|$$

Fromm Birkhoff's theorem

$$\boxed{\Lambda^{-1} = \left| \ln \frac{\int \mathrm{d}\mathbb{P}(\omega)\, |t(\omega)|}{\int \mathrm{d}\mathbb{P}(\omega)\, |m(\omega)|} \right|}$$

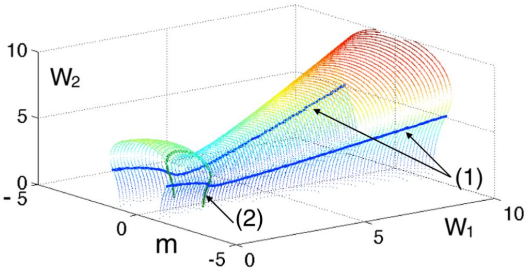
[Mondragon et al, Phys. Rev. Lett. (2014)]

A typical example

White noise disorder:

$$\omega = [-\tfrac{1}{2}, \tfrac{1}{2}]^{\mathbb{Z}}, \quad \mathrm{d}\mathbb{P}(\omega) = d\omega, \quad t(\{\omega_x\}) = 1 + W_1\,\omega_0, \quad m(\{\omega_x\}) = m + W_2\,\omega_0$$

$$\Lambda^{-1} = \left| \ln\left[\frac{|2+W_1|^{1/W_1+1/2}}{|2-W_1|^{1/W_1-1/2}} \frac{|2m-W_2|^{m/W_2-1/2}}{|2m+W_2|^{m/W_2+1/2}} \right] \right|$$



Spectacular phenomenon:

The emergence of a manifold of zero Lyapunov exponent at very high levels of disorder.

It implies a sudden insulator/metal phase transition in dynamics of quasi 1-dimensional chain.

TOPOLOGICAL MATTER *Science* **362**, 929–933 (2018)

Observation of the topological Anderson insulator in disordered atomic wires

Eric J. Meier[1], Fangzhao Alex An[1], Alexandre Dauphin[2], Maria Maffei[2,3], Pietro Massignan[2,4*], Taylor L. Hughes[1*], Bryce Gadway[1*]

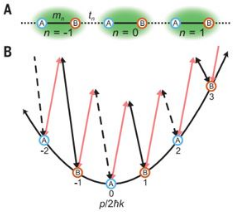


Fig. 1. Synthetic chiral symmetric wires engineered with atomic momentum states.

Computing Pairings with Cyclic Cocycles at Strong Disorder

The generic models on $\mathbb{C}^N \otimes \ell^2(\mathbb{Z}^d)$:

$$H_\omega = \sum_{q\in\mathbb{Z}^d} S_q \sum_{x\in\mathbb{Z}^d} w_q(\tau_x\omega) \otimes |x\rangle\langle x|$$

come from $C(\Omega) \rtimes_\tau \mathbb{Z}^d$ via the GNS rep corresponding to the state $\mathcal{T}_\omega(\sum_q w_q u_q) = w_q(\omega)$.

This algebra comes with:

- A standard differential calculus

$$\mathcal{T}(\textstyle\sum_q w_q u_q) = \int \mathrm{d}\mathbb{P}(\omega)\, w_0(\omega), \quad \partial_i = \imath \sum_q q_i\, w_q u_q, \quad i = 1, \ldots, d.$$

- A standard finite-volume approximation (which carries the differential calculus!)

$$\mathbb{Z} \mapsto \mathbb{Z}_N, \quad \Omega \mapsto \Omega_N, \ \tau_x^{\circ N}(\omega) = \omega \ \ \forall\, x \in \mathbb{Z}^d \text{ and } \omega \in \Omega_N$$

  There is an epi-morphism of C^*-algebras:

$$\mathfrak{p}_N : C(\omega) \rtimes_\tau \mathbb{Z}^d \to C(\Omega_N) \rtimes_\tau \mathbb{Z}^d_N, \quad \mathfrak{p}\Big(\sum_q w_q u_q\Big) = \sum_q w_q|_{\Omega_N} u^{(N)}_{q \bmod N}$$

Theorem [E. P. 2013, 2016] (some of the assumptions are not shown here)

Let h be a smooth element from $\in C(\Omega) \rtimes_\tau \mathbb{Z}^d$ and $\hat{h} = \mathfrak{p}_N(h)$ etc. Then

$$\lim_{N\to\infty} N^m \left| \mathcal{T}\big(\partial^{\alpha_1} G_1(h) \ldots \partial^{\alpha_n} G_n(h)\big) - \widehat{\mathcal{T}}\big(\hat{\partial}^{\alpha_1} G_1(\hat{h}) \ldots \hat{\partial}^{\alpha_n} G_n(\hat{h})\big) \right| \leq \infty, \quad \forall\ m > 1.$$

The estimates also hold if h is taken from appropriate Sobolev spaces.

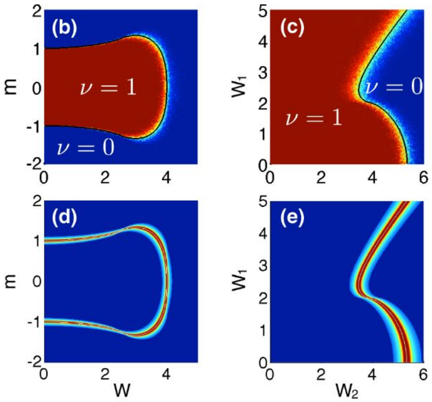
Example: For models with chiral symmetry in odd dimension:

$$\mathrm{sign}(h) = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}$$

and we can compute the pairing of u and an odd cyclic-cocycle

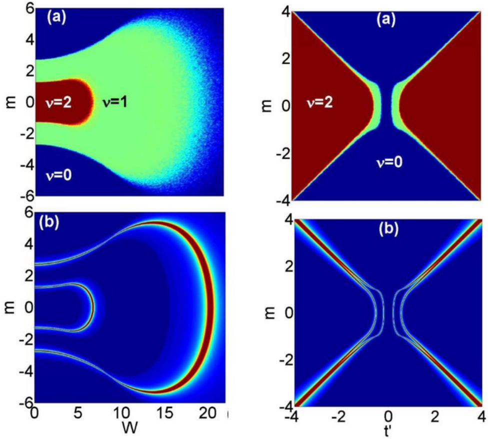
$$\nu_d(u) = \Lambda_d \sum_\sigma (-1)^\sigma \mathcal{T}\Big(\prod_{j=1}^{d} u^{-1} \partial_{\sigma_j} u\Big).$$

Numerical Results for the 1-Dimensional Model [Mondragon, Phys. Rev. Lett. 2014]



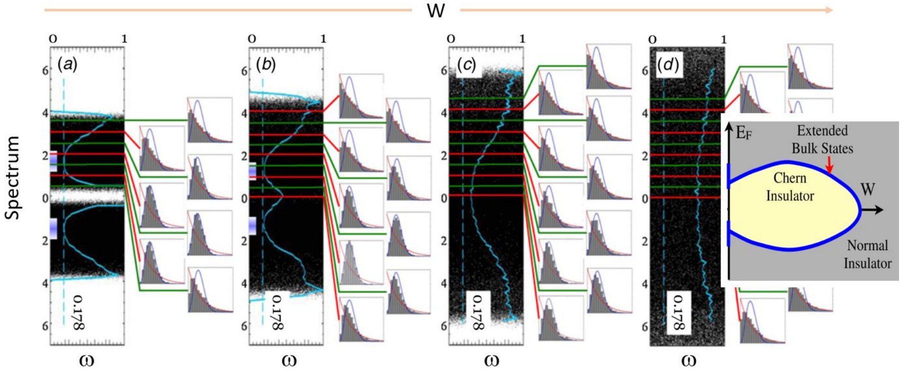
The amazing fact is that, since $0 \in \mathrm{Spec}(h)$, $u \notin C(\Omega) \rtimes_\tau \mathbb{Z}^d$.

$$(H\psi)_x = m_x\,\hat{\sigma}_2\psi_x + {}^1\!/\!{}_2\, t_x[(\hat{\sigma}_1 + i\hat{\sigma}_2)\psi_{x+1} + (\hat{\sigma}_1 - i\hat{\sigma}_2)\psi_{x-1}] + {}^1\!/\!{}_2\, t'[(\hat{\sigma}_1 + i\hat{\sigma}_2)\psi_{x+2} + (\hat{\sigma}_1 - i\hat{\sigma}_2)\psi_{x-2}],$$



Numerical Results for a 2-Dimensional Model (honeycomb lattice) [E.P., J. Phys. A (2011)]

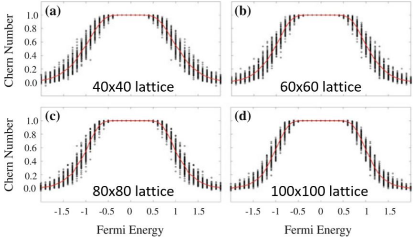
$$H_\omega = \sum_{\langle x,y\rangle} |x\rangle\langle y| + 0.6\imath \sum_{\langle\langle x,y\rangle\rangle} \big(|x\rangle\langle y| - |y\rangle\langle x|\big) + W \sum_x \omega_x |x\rangle\langle x|, \quad \big(\langle,\rangle / \langle\langle,\rangle\rangle = \text{first/second neighbors}\big)$$

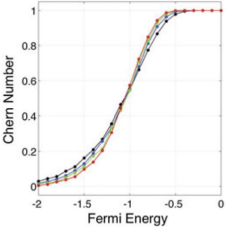


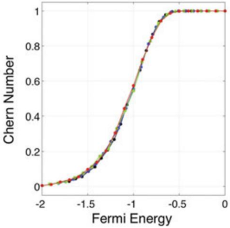
A manifold of critical extended states develops again.

Computation of Even Pairings

$$\mathrm{Ch}_d(p_E) = \Lambda_d \sum_{\sigma} (-1)^{\sigma} \mathcal{T}\Big(p_E \prod_{j=1}^{N} \partial_{\sigma_j} p_E \Big), \quad p_E = \chi_{(-\infty,E]}(h).$$

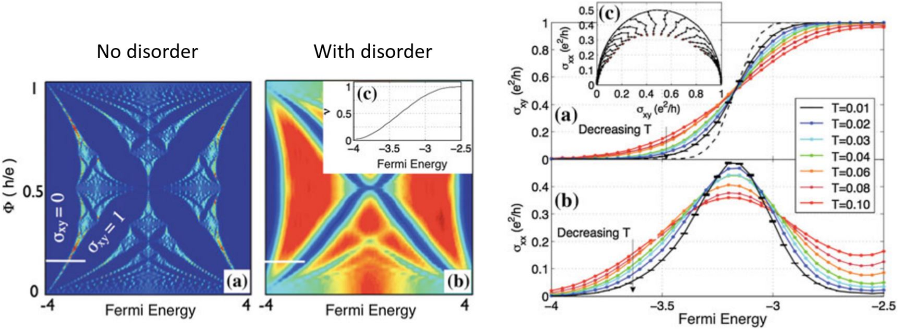






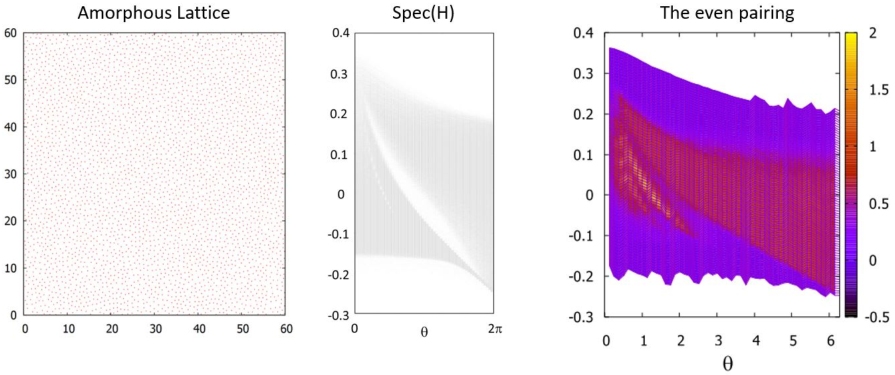
Fermi energy	40 × 40	60 × 60	80 × 80	100 × 100
−0.7000000000000000	0.8672349391630054	0.9079791895178040	0.9301639459675241	0.9432234611493278
−0.6000000000000000	0.9392873717233425	0.9636994770114942	0.9802652381114992	0.9872940741308633
−0.5000000000000000	0.9784417158133359	0.9935074963179980	0.9974987656403326	0.9988846769813913
−0.4000000000000000	0.9958865415757685	0.9992024708366942	0.9998527876642247	0.9999656328302596
−0.3000000000000000	0.9998184404341747	0.9999824660477071	0.9999988087144891	0.9999996457562911
−0.2000000000000000	0.9999952917010211	0.9999977443008894	0.9999999997655000	0.9999999999862120
−0.1000000000000000	0.9999999046002306	0.9999999998972079	0.9999999999998473	0.9999999999999849
0.0000000000000000	0.9999999963422543	0.9999999999988873	0.9999999999999996	0.9999999999999999

- $h \in C^*(C(\Omega), u_1, u_2), \quad u_1 u_2 = e^{2\pi\imath\Phi} u_2 u_1, \quad f u_j = u_j(f \circ \tau_j).$
- $h = u_1 + u_1^* + u_2 + u_2^* + W f, \quad f(\{\omega_{x,y}\}) = \omega_{0,0}.$

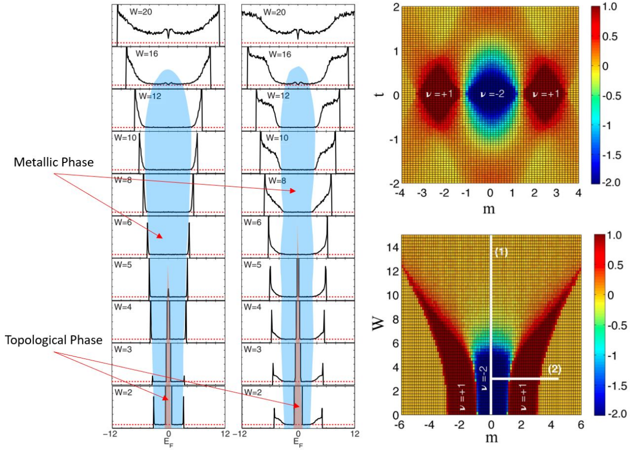


An Amorphous System under Magentic Field [Bourne et al, J. Phys. A (2018)]

- $H : \ell^2(\mathcal{L}) \to \ell^2(\mathcal{L}), \quad H = \sum_{x,x' \in \mathcal{L}} e^{\imath\theta\, x \wedge x'} e^{-3|x-x'|} |x\rangle\langle x'|$
- H can be generated from a grupoid algebra canonically associated to $\mathcal{L}$.

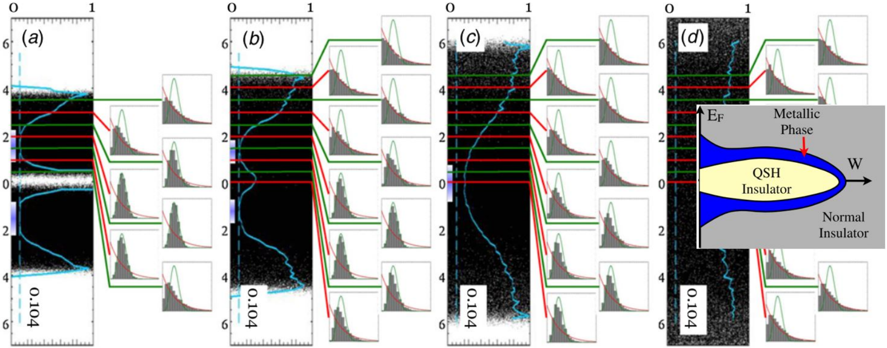


- $C(\Omega) \rtimes \mathbb{Z}^3 \ni h = \frac{1}{2\imath} \sum_{j=1}^{3} \Gamma_j \otimes (u_j - u_j^*) + \Gamma_4 \otimes \Big[M + \frac{1}{2} \sum_{j=1} (u_j + u_j^*) + t\imath \Gamma_1 \Gamma_3 + W\, f \Big]$



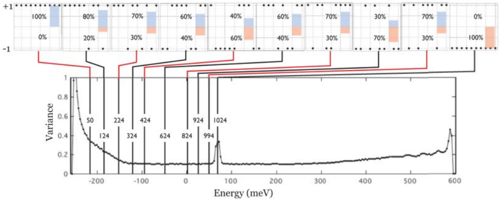
- System from AII class (time-reversal symmetric) in $d = 2$

$$H_0^{\rm QSH} = \sum_{\langle nm\rangle,\sigma} |\boldsymbol{n},\sigma\rangle\langle \boldsymbol{m},\sigma| + \sum_{\langle\langle nm\rangle\rangle,\sigma} \alpha_n (t/2 + \mathrm{i}\eta[\hat{S}\cdot \underline{\mathbf{d}_{km}\times\mathbf{d}_{nk}}]_{\sigma,\sigma})|\boldsymbol{n},\sigma\rangle\langle \boldsymbol{m},\sigma| + \mathrm{i}\lambda \sum_{\langle nm\rangle,\sigma\sigma'} [\mathbf{e}_z\cdot(\hat{S}\times\underline{\mathbf{d}_{nm}})]_{\sigma,\sigma'}|\boldsymbol{n},\sigma\rangle\langle \boldsymbol{m},\sigma'|.$$



- A system from AII class (time-reversal symmetry) in $d = 3$

Evaluation of the Z2-invariant at strong disorder

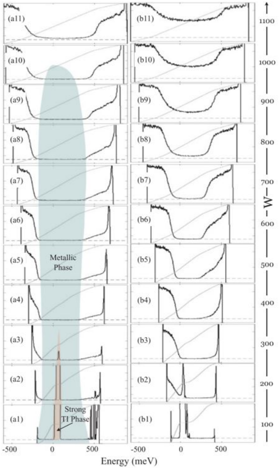


For the cases with time-reversal and/or particle-hole symmetries the relevant pairing is

$$KKO(C\ell_{j,0}, C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d) \times KKO(C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d \hat{\otimes} C\ell_{0,d}, C(\Omega)) \to KKO(C\ell_{j,d}, C(\Omega))$$

Generating a local formula for this Clifford index is still open problem.

Bourne et al, AHP (2017)



The Conjectured Topological Classification Table

j	TRS	PHS	CHS	CAZ	0, 8	1	2	3	4	5	6	7
0	0	0	0	A	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
1	0	0	1	AIII		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
0	$+1$	0	0	AI	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	$+1$	$+1$	1	BDI	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
2	0	$+1$	0	D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
3	-1	$+1$	1	DIII		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
4	-1	0	0	AII	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
5	-1	-1	1	CII		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
6	0	-1	0	C			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
7	$+1$	-1	1	CI				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

- A. P. Schnyder, S. Ryu, A. Furusaki, A. W. W. Ludwig, *Classification of topological insulators and superconductors in three spatial dimensions*, Phys. Rev. **B 78**, 195125 (2008).
- A. Kitaev, *Periodic table for topological insulators and superconductors*, (Advances in Theoretical Physics: Landau Memorial Conference) AIP Conference Proceedings **1134**, 22-30 (2009).
- S. Ryu, A. P. Schnyder, A. Furusaki, A. W. W. Ludwig, *Topological insulators and superconductors: tenfold way and dimensional hierarchy*, New J. Phys. **12**, 065010 (2010).

Part 2

Index Theorems for the cyclic cocycles to explain

- The quantization of the pairings in the extreme disorder regime
- The emergence of the critical manifolds of extended states

[Following here E.P, Leung, Bellissard, J. Phys. A (2013)]

The Setting (d = even)

The C^*-algebra and its differential calculus

- $(\Omega, \tau, \mathbb{Z}^d, \mathrm{d}\mathbb{P})$ ergodic dynamical system
- $\mathcal{A} = C(\Omega) \rtimes \mathbb{Z}^d$, $d = \text{even}$ $\quad \left(\mathcal{A} \ni a = \sum_{q \in \mathbb{Z}^d} a_q u_q,\ a_q \in C(\Omega) \right)$
- $\pi_\omega(a) = \sum_q \left(\sum_x a_q(\tau_x \omega)\, |x\rangle\langle x| \right) S_q$ $\quad$ ($\mathbb{P}$-almost sure faithful representations on $\ell^2(\mathbb{Z}^2)$)
- $\partial_j a = \imath \sum_q q_i\, a_q u_q$, $\quad \mathcal{T}(a) = \int \mathrm{d}\mathbb{P}(\omega)\, a_0(\omega)$

Initial data (a gentle start):

- $h \in \mathcal{A}^\infty$, $\quad h = h^*$
- $G \subset \mathbb{R} \setminus \mathrm{Spec}(h) \neq \emptyset$ (spectral gap condition)
- $p_E = \chi_{(-\infty, E]}(h)$, $E \in G$ (spectral projection) [In general, E is fixed by the electron density]

The object of interest

$$\xi(a_0, a_1, \ldots, a_d) = \Lambda_d \sum_{\sigma \in \mathcal{S}_d} \mathcal{T}\Big(a_0 \prod_{j=1}^{d} \partial_{\sigma_j} a_j\Big)$$

Key properties (following directly from $\mathcal{T}(\partial_j a) = 0$):

- It is (even) cyclic

$$\xi(a_1, \ldots, a_d, a_0) = \xi(a_1, \ldots, a_d, a_0)$$

- It is closed $b\xi = 0$ against the Hochschild coboundary map

$$(b\xi)(a_0, a_1, \ldots, a_d, a_{d+1}) = \sum_{j=0}^{d} (-1)^j \xi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{d+1}) - \xi(a_{d+1} a_0, \ldots, a_d)$$

As a result (Connes 1985):

- There exists a pairing with the K_0-classes landing in a countable subgroup of the real axis:

$$\big\langle [\xi], [p]_0 \big\rangle := \xi(p, \ldots, p)$$

Physical Content

When applied to the spectral projection p_E (ϕ_{ij} = magnetic flux through the (ij)-facet):

- If $d = 2$ (Bellissard et al 1994):

$$\xi(p_E, p_E, p_E) = \sigma_H \quad \text{(the Hall conductance at zero temperature)}$$

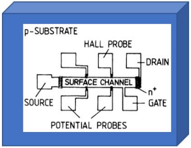
- If $d \geq 4$ (E. P. and Schulz-Baldes 2016):

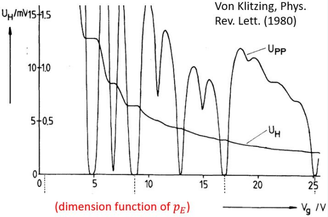
$$\xi(p_E, \ldots, p_E) = \partial_{\phi_{i_1,i_2}} \ldots \partial_{\phi_{i_{d-1},i_d}} \sigma_H \quad \text{(non-linear transport coefficient)}$$

Although the pairing assures the quantization of these transport coefficients, the experimental reality is harsh. Clearly, *E* is located in the essential spectrum!!!

$$\rho_H = \frac{U_H}{I} \approx 1/\sigma_H$$

$$R_D = \frac{U_{PP}}{I}$$





Von Klitzing, Phys. Rev. Lett. (1980)

Time to Examine the Domain of the Co-Cycle

In the standard approach

- $\mathcal{D}(\xi) = \mathcal{A}^\infty$ (defined by the semi-norms $\|\partial^\alpha a\|$).

However, Hölder inequality gives:

- $|\xi(a_0, a_1, \dots, a_d)| \leq \|a_0\|_\infty \prod_{j=1}^d \left(\sum_{k=1}^d \|\partial_k a_j\|_d \right), \quad \|a\|_p = \left[\mathcal{T}(|a|^p)\right]^{\frac{1}{p}}$

- $|\xi(a_0, a_1, \dots, a_d) - \xi(a_0', a_1', \dots, a_d')| \leq \text{Factor} \times \sum_{j=0}^d \left(\sum_{k=1}^d \|\partial_k (a_j - a_j')\|_d \right)$

Important conclusion:

The natural domain for ξ is the Sobolev space $\mathcal{W}_{1,d}(\mathcal{A}, \mathcal{T})$ defined by the norm

$$\|a\|_S = \|a\|_\infty + \sum_{k=1}^d \left[\mathcal{T}\left(|\partial_k a_j|^p\right)\right]^{\frac{1}{p}}$$

Furthermore, $p_E \in \mathcal{W}_{1,d}(\mathcal{A}, \mathcal{T})$ in the experimental harsh setting.

Quantized Calculus

The tuple $\left(\eta_\omega : \mathcal{A} \to \mathbb{B}(\mathcal{H}), \widehat{D}_{x_0} = \frac{D_{x_0}}{|D_{x_0}|}, \Gamma_0\right)$ is an even Fredholm module, where

- $\mathcal{H} = \mathbb{C}^{2^d} \otimes \ell^2(\mathbb{Z}^d), \quad \eta_\omega = 1 \otimes \pi_\omega$
- Γ_i = Clifford matrices and $\Gamma_0 = -i^n \Gamma_1 \cdots \Gamma_d$
- $D_{x_0} = \sum_{i=1}^d \Gamma_i \otimes (X_i - x_0)$

If the module is $(d, \infty) -$ *summable*, then Connes-Chern character comes into play:

$$\mathrm{Tr}_{\mathrm{s}}\left(\Gamma_0 \left[\widehat{D}_{x_0}, \eta_\omega(p_E)\right]^d\right) = \mathrm{Ind}\left(\eta_\omega^-(p_E)\, \widehat{D}_{x_0}\, \eta_\omega^+(p_E)\right)$$

Note, however, that we need to push into the Sobolev setting.

$\mathbb{P}$-Almost Sure Summability

For $a \in \mathcal{W}_{1,d}(\mathcal{A},\mathcal{T})$, the following identity holds $\mathbb{P}$-almost surely ($\Gamma(\hat{x}) = \Gamma - \hat{x}\,(\hat{x}\cdot\Gamma)$):

$$\mathrm{Tr}_{\mathrm{Dix}}\Big(\big(\imath[\widehat{D}_{x_0},\eta_\omega(a)]\big)^d\Big) = \tfrac{1}{d}\int_{S_{d-1}} \mathrm{d}\hat{x}\ \mathrm{tr}_\Gamma \otimes \mathcal{T}\Big(\big(\Gamma(\hat{x})\cdot\nabla(a)\big)^d\Big)$$

Corollary: $\mathbb{P}$-almost surely, the module

$$\Big(\eta_\omega : L^\infty(\mathcal{A}) \to \mathbb{B}(\mathcal{H}), \widehat{D}_{x_0} = \frac{D_{x_0}}{|D_{x_0}|}, \Gamma_0\Big)$$

is (d,∞)-summable over $\mathcal{W}_{1,d}(\mathcal{A},\mathcal{T})$. As a result, $\mathbb{P}$-almost surely,

$$\mathrm{Tr}_{\mathrm{s}}\Big(\Gamma_0\big[\widehat{D}_{x_0},\eta_\omega(p_E)\big]^d\Big) = \mathrm{Ind}\Big(\eta_\omega^-(p_E)\,\widehat{D}_{x_0}\,\eta_\omega^+(p_E)\Big) \in \mathbb{Z}.$$

Using quantized calculus, we produced a family $\mathbb{Z}$-valued cyclic co-cycles over $\mathcal{W}_{1,d}(\mathcal{A},\mathcal{T})$.

Local Formula for the Connes-Chern Character

Connes-Moscovici local-index formula:

$$\text{Index } PUP =$$

$$\sum_{n\leq p}(-1)^{\frac{n-1}{2}}\left(\frac{n-1}{2}\right)!\sum_{k,q}\frac{(-1)^{|k|}}{k_1!\dots k_n!}\,\alpha_k\,\frac{1}{q!}\,\sigma_{m-q}(m)\,\mathrm{Res}_{s=0}\,s^q\,\zeta_{(k,n)}(s)$$

(see C. Bourne's PhD thesis)

However, in our particular context, we can use some remarkable identities

- In $d=2$, the identity is due to Alain Connes (1985)

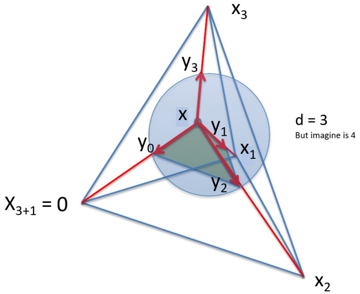
$$\sum_q\left(1-\frac{x\overline{(x+x_1)}}{|q(q+x_1)|}\right)\left(1-\frac{(x+x_1)\overline{(x+x_2)}}{|(x+x_1)(q+x_2)|}\right)\left(1-\frac{(x+x_2)\overline{q}}{|(x+x_2)x|}\right)=2\pi\imath\,x_1\wedge x_2$$

- The generalization to $d>2$ looks like this ($\boldsymbol{x}_{d+1}=\boldsymbol{0}$)

$$\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{x}\ \mathrm{tr}\Big\{\Gamma_0\prod_{i=1}^{d}\Big(\frac{\boldsymbol{\Gamma}\cdot(\boldsymbol{x}_i+\boldsymbol{x})}{|\boldsymbol{\Gamma}\cdot(\boldsymbol{x}_i+\boldsymbol{x})|}-\frac{\boldsymbol{\Gamma}\cdot(\boldsymbol{x}_{i+1}+\boldsymbol{x})}{|\boldsymbol{\Gamma}\cdot(\boldsymbol{x}_{i+1}+\boldsymbol{x})|}\Big)\Big\}=\frac{(2\imath\pi)^{d/2}}{(d/2)!}\sum_{\rho\in\mathcal{S}_d}(-1)^{\rho}\prod_{i=1}^{d}x_{i,\rho_i}$$

Sketch of Proof of the Geometric Identity

- Left side generates terms like $\mathrm{tr}\Big\{\Gamma_0 \prod_{i=1}^{d} \boldsymbol{\Gamma}\cdot \boldsymbol{y}_{\alpha_i}\Big\} \;=\; (2\imath)^{d/2} d!\ \mathrm{Vol}[\mathbf{0}, \boldsymbol{y}_{\alpha_1}, \ldots, \boldsymbol{y}_{\alpha_d}]$



- $\int \mathrm{d}\boldsymbol{x} \sum_{\{\alpha_1,\ldots,\alpha_d\}} \mathrm{Vol}\big[\mathbf{0}, \boldsymbol{y}_{\alpha_1}, \ldots, \boldsymbol{x}_{\alpha_d}\big] = \mathrm{Vol}(\text{unit ball}) \times \mathrm{Vol}\big(\mathbf{0}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_d\big)$
- The volume on the right is expressed as a determinant. □

Theorem: For any $p \in \mathcal{W}_{1,d}(\mathcal{A},\mathcal{T})$, $\mathbb{P}$-almost surely

- $\eta^-_\omega(p)\widehat{D}_{x_0}\eta^+_\omega(p)$ is a Fredholm operator.
- $\mathrm{Ind}\Big(\eta^-_\omega(p)\widehat{D}_{x_0}\eta^+_\omega(p)\Big) = \xi(p,\dots,p)$

If $p(t) \in \mathcal{W}_{1,d}(\mathcal{A},\mathcal{T})$ varies continuously w.r.t. the norm $\sum_{k=1}^d \|\partial_k(\cdot)\|_d$, then

- $\xi\big(p(t),\dots,p(t)\big) = \mathrm{constant} \in \mathbb{Z}$.

Proof:

- $\eta^-_\omega(p)\widehat{D}_{x_0}\eta^+_\omega(p) - \eta^-_{\tau_x\omega}(p)\widehat{D}_{x_0}\eta^+_{\tau_x\omega}(p) =$ compact operator
- $\eta^-_\omega(p)\widehat{D}_{x_0}\eta^+_\omega(p) - \eta^-_\omega(p)\widehat{D}_{x_0'}\eta^+_\omega(p) =$ compact operator
- $\mathrm{Ind}\Big(\eta^-_\omega(p_E)\,\widehat{D}_{x_0}\,\eta^+_\omega(p_E)\Big) = \int \mathrm{d}\mathbb{P}(\omega) \int \mathrm{d}x_0\ \mathrm{Tr}_\mathrm{s}\Big(\Gamma_0\big[\widehat{D}_{x_0},\eta_\omega(p_E)\big]^d\Big)$
- Evaluate the right side using the geometric identity.

The Index Theorem for Odd Dimensions [E.P, Schulz-Baldes J. Func. Anal. (2016)]

Theorem: For any $u \in \mathcal{W}_{1,d}(\mathcal{A}, \mathcal{T})$, $\mathbb{P}$-almost surely

- $E_{x_0} \eta_\omega(u) E_{x_0}$ ($E_{x_0} = \chi_{(-\infty,0]}(D_{x_0})$) is a Fredholm operator.
- $\mathrm{Ind}\Big(E_{x_0} \eta_\omega(u) E_{x_0}\Big) = \xi(u^{-1}, u, \ldots, u)$

If $u(t) \in \mathcal{W}_{1,d}(\mathcal{A}, \mathcal{T})$ varies continuously w.r.t. the norm $\sum_{k=1}^d \|\partial_k(\cdot)\|_d$, then

- $\xi\big(u(t)^{-1}, \ldots, u(t)\big) = \mathrm{constant} \in \mathbb{Z}$.

$$(H\boldsymbol{\psi})_x = m_x\, \hat{\sigma}_2 \boldsymbol{\psi}_x + 1/2\, t_x[(\hat{\sigma}_1 + i\hat{\sigma}_2)\boldsymbol{\psi}_{x+1} + (\hat{\sigma}_1 - i\hat{\sigma}_2)\boldsymbol{\psi}_{x-1}] + 1/2\, t'[(\hat{\sigma}_1 + i\hat{\sigma}_2)\boldsymbol{\psi}_{x+2} + (\hat{\sigma}_1 - i\hat{\sigma}_2)\boldsymbol{\psi}_{x-2}],$$