Twisted differential geometry and dispersion relations in κ-deformed cosmology

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Noncommutative Geometry and Physics Seminar series

Motivation

Quantum Gravity

$\downarrow$

Quantum Geometry

$\downarrow$

Classical Geometry

- Noncommutative geometry - generalised notion of geometry
- The noncommutative nature allows for obtaining quantum gravitational corrections to the classical solutions.

- Noncommutative geometry - generalised notion of geometry
- The noncommutative nature allows for obtaining quantum gravitational corrections to the classical solutions.
- Can be helpful in providing the phenomenological models quantifying the effects of quantum gravity.
- One of the mostly studied possible phenomenological effects of quantum gravity is the **modification in wave dispersion**. Such investigations were inspired by the observations of gamma ray bursts (GRBs).
- Noncommutative differential geometry based on Drinfeld twist deformation canonically gives NC wave equations - A top-down geometric approach to dispersion relations

Differential Geometry vs NC Differential Geometry

M - manifold and $C^\infty(M) = A$ - functions on a manifold

$\rightarrow A_\star$ - NC deformation of the algebra of functions on a manifold

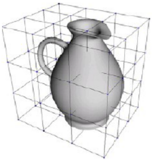
and

Ω^1 space of 1-forms, e.g. differentials:

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial x^\mu} \mathrm{d}x^\mu$$

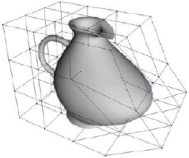
$$f\mathrm{d}g = (\mathrm{d}g)f$$

$\rightarrow$ noncommutative differential structure:
differential bimodule (Ω^1, d) of 1-forms with d - obeying the Leibniz rule and
$\rightarrow f \star \mathrm{d}g \neq (\mathrm{d}g) \star f$



Quantum deformations

Twist deformation



Classical —— deformation = quantization ——→ Noncommutative (quantum)

Symmetry

Lie Algebra | Hopf Algebra (Quantum Group)

$$g \longrightarrow H = (Ug, \Delta_0, S_0, \epsilon) \xrightarrow[\mathcal{F} \in Ug \otimes Ug]{} (Ug^{\mathcal{F}}, \Delta^{\mathcal{F}}, S^{\mathcal{F}}, \varepsilon)$$

Space-time

$$\mathcal{A} = (C^\infty(M), \mu)$$

$$[x^\mu, x^\nu] = 0$$

$$\xrightarrow[\mathcal{F} \in Ug \otimes Ug]{x^\mu \to \hat{x}^\mu}$$

$$\mathcal{A}^{\mathcal{F}} = (C^\infty(M), \star)$$

$$x^\mu \star x^\nu = \mu \circ \mathcal{F}^{-1}(x^\mu \otimes x^\nu)$$

$$[x^\mu, x^\nu] = x^\mu \star x^\nu - x^\nu \star x^\mu$$

Lie algebra of vector fields as Hopf algebra

- **Deformations of spacetime symmetries** - Lie algebra g of vector fields ξ

- Ug as Hopf algebra $H = (Ug, \Delta_0, \epsilon, S_0)$
- In the coordinate basis $\xi \in g$: $\xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \xi^\mu \partial_\mu$.

- This algebra generates the diffeomorphism symmetry; one can also consider subalgebras of g like Poincaré algebra or conformal algebra as symmetry.

Twisting

$$(H, \mathcal{A}) \rightleftarrows (H^{\mathcal{F}}, \mathcal{A}_\star)$$

The twist $\mathcal{F}$ is an invertible element of $H \otimes H$.

$$\mathcal{F} = 1 \otimes 1 + \mathcal{O}(h),$$

which provides an undeformed case at the zero-th order in the **deformation parameter** h.

Notation:

$$\mathcal{F} = \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha, \quad \mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha,$$

(sum over $\alpha = 1, 2, \ldots\infty$ assumed)
$\bar{\mathrm{f}}^\alpha \in H$ and $\mathrm{f}^\alpha \in H$

Quantum spacetime: star-product

$$A = (C^\infty(M), \cdot) \qquad \Longrightarrow \qquad A_\star = (C^\infty(M), \star)$$

the algebra of smooth functions becomes a **noncommutative spacetime** with the twisted $\star$-product

$$x^\mu \star x^\nu \;=\; \cdot \;\; \mathcal{F}^{-1}(x^\mu \otimes x^\nu) = \bar{\mathrm{f}}^\alpha(x^\mu)\bar{\mathrm{f}}_\alpha(x^\nu)$$

$x^\mu, x^\nu \in C^\infty(M)$.

- such $\star$-product is noncommutative and associative.
- $A^\mathcal{F}$ can be represented by deformed, $\star$-commutators of noncommutative coordinates:

$$[x^\mu, x^\nu] = x^\mu \star x^\nu - x^\mu \star x^\mu$$

Quantum (noncommutative) spacetimes

1. Canonical (Moyal-Weyl) spacetime A_θ:
   $$[x^\mu, x^\nu] = ih\theta^{\mu\nu}$$
   with deformation parameter h of length^2 (L_P) dim.

   S. Doplicher, K. Fredenhagen, J. E. Roberts, Commun. Math. Phys. 172 (1995), [arXiv:hep-th/0303037].

2. Lie-algebraic type spacetime:
   $$[x^\mu, x^\nu] = ih\theta^{\mu\nu}_\rho x^\rho$$
   with deformation parameter h of mass (M_P) dim.

   Special case: A_κ

   $$[x^0, x^k] = \frac{i}{\kappa} x^k \quad , \quad [x^i, x^k] = 0$$

   \- the so-called: κ-Minkowski spacetime.

   S. Majid, H. Ruegg Phys.Lett. B334 (1994) [hep-th/9405107] ; S. Zakrzewski J. Phys. A 127 (1994).

Drinfeld Twists - examples

The **canonical (Moyal-Weyl) noncommutative spacetime** can be obtained by twist deformation:

$$\mathcal{F} = \exp\left(-\frac{i}{2} h\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu\right)$$

The twist has **support in the Poincaré algebra**, i.e. $\mathcal{F} \in \mathcal{U}_{\mathfrak{iso}(1,n-1)} \otimes \mathcal{U}_{\mathfrak{iso}(1,n-1)}$. (The minimal algebra $\mathcal{F} \in \mathcal{U}_{\mathfrak{t}^n} \otimes \mathcal{U}_{\mathfrak{t}^n}$.)

The Moyal-Weyl $\star$-product of functions on $\mathbb{R}^n$:

$$f \star g = e^{\frac{i}{2} h\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}} f(x)g(y)|_{x=y} = f(x)g(x) + \frac{i}{2} h\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} f \frac{\partial}{\partial x^\nu} g + ...$$

giving

$$[x^\mu, x^\nu] = x^\mu \star x^\nu - x^\mu \star x^\mu = ih\theta^{\mu\nu}$$

Drinfeld Twists - examples

[*J. Lukierski, V. Lyakhovsky, M. Mozrzymas, Phys.Lett. B538 (2002)*
A. Borowiec, A.P, EPJ C 74, 3 (2014)]

The **light-cone κ-Minkowski spacetime** can be implemented by the **extended Jordanian twist**

$$\mathcal{F} \quad = \quad \exp\left(\frac{i}{\kappa}(x_+\partial_a - x_a\partial_+)\otimes\partial_a\right)\exp\left(i(x_+\partial_- - x_-\partial_+)\otimes\ln(1+\frac{1}{\kappa}\partial_+)\right)$$

The twist has support in **the (null-plane) Poincare algebra**. (leading to the 'Null-Plane Quantum Poincare Algebra' *[A. Ballesteros, F. J. Herranz, M. A. del Olmo, M. Santander, PLB351'95]*)

- giving the light-cone κ-Minkowski spacetime:
  $[x^\pm, x^a] = \pm\frac{i}{\kappa}x^\pm \quad , \quad [x^a, x^b] = 0 \quad , \quad [x^+, x^-] = \frac{i}{\kappa}(x^+ - x^-)$

Drinfeld Twists - examples

[*Jong-Gepn Bu, Hyeong-Chan Kim, Youngone Lee, Chang Hyon Vac, Jae Hyung Yee, Phys. Lett. B665 (2008), [arXiv:hep-th/0611175]*
A. Borowiec, A.P. SIGMA 6 (2010), 086 [arXiv:1005.4429]]

The κ-**Minkowski spacetime** can be implemented by the **Abelian twist**

$$\mathcal{F} = \exp\left[-\frac{i}{2\kappa}\left(\partial_0 \otimes x^k\partial_k - x^k\partial_k \otimes \partial_0\right)\right] \qquad (1)$$

The smallest subalgebra generated by $D = x^k\partial_k, P_0 = -i\partial_0$ and the Lorentz generators turns out to be entire $igl(n)$ **algebra**.

$$f \star g = e^{\frac{i}{2\kappa}\left(\frac{\partial}{\partial x^0} y^k \frac{\partial}{\partial y^k} - x^k \frac{\partial}{\partial x^k}\frac{\partial}{\partial y^0}\right)} f(x)g(y)|_{x=y}$$

giving:

$$[x^0, x^k] = \frac{i}{\kappa}x^k \quad , \quad [x^i, x^k] = 0$$

Drinfeld Twists - examples

[*A. Borowiec, A.P, Phys.Rev.D79 (2009) [arXiv:0812.0576]*]

The κ-**Minkowski spacetime** can be implemented by the **Jordanian twist**

$$\mathcal{F} = \exp\left(-x^\mu \partial_\mu \otimes \ln(1 - \frac{i}{\kappa}\partial_0)\right)$$

The twist has support in U_{pw} of **the Poincaré-Weyl algebra** $pw = span\{M_{\mu\nu}, P_\mu, D = -ix^\mu \partial_\mu\}$.

$$f \star g = \exp(x^\mu \frac{\partial}{\partial x^\mu} \otimes \ln(1 - \frac{i}{\kappa}\frac{\partial}{\partial y^0}))f(x)g(y)|_{x=y}$$

giving:

$$[x^0, x^k] = \frac{i}{\kappa}x^k \quad , \quad [x^i, x^k] = 0$$

Twisted differential geometry

- Noncommutative (twisted) differential geometry approach is based on Drinfeld twist $\mathcal{F}$ deformation.
- Can be implemented for any twist $\mathcal{F}$ and any curved background (g).
- Noncommutative differential geometry based on Drinfeld twist deformation canonically gives **NC wave equations** - Allows for a study of the corresponding dispersion relations

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- Aim: wave equation for the Jordanian twist - giving k-Minkowski spacetime - in the presence of a FLRW cosmological background & dispersion relations [*P. Aschieri, A. Borowiec, A.P., JCAP 04 (2021) [arXiv:2009.01051]]*.

Twisted differential calculus

[*P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp , J. Wess, Class. Quant. Grav. 22 (2005) [arXiv:hep- th/0504183]*
P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess, Class. Quant. Grav. 23 (2006) [arXiv:hep-th/0510059]]

Take:

- Algebra A (of smooth functions on spacetime M) and the action of the Lie algebra g on A via the Lie derivative.
- and the algebra: $\Omega^\bullet = A \oplus \Omega^1 \oplus \Omega^2 \oplus ...$ of exterior forms on M.

Then:

- Twist deform this to $A_\star$ and $\Omega^\bullet_\star$ (same as $\Omega^\bullet$ as a vector space) with the new product

$$\omega \wedge_\star \omega' = \bar{\mathrm{f}}^\alpha(\omega) \wedge \bar{\mathrm{f}}_\alpha(\omega')$$

with the action of $\bar{\mathrm{f}}_\alpha$ on ω via the Lie derivative along the vector fields defining $\mathcal{F}^{-1}$.
If $\omega = f \in C^\infty(M)$:

$$f \star \omega' = \bar{\mathrm{f}}^\alpha(f)\bar{\mathrm{f}}_\alpha(\omega')$$

- The Lie derivative commutes with the exterior derivative: the usual (undeformed) exterior derivative satisfies **the Leibniz rule**

$$\mathrm{d}(f \star g) = \mathrm{d}f \star g + f \star \mathrm{d}g,$$

  and

$$\mathrm{d}^2 = 0,$$

- for forms of homogeneous degree $\omega \in \Omega^r$,

$$\mathrm{d}(\omega \wedge_\star \omega') = \mathrm{d}\omega \wedge_\star \omega' + (-1)^r \omega \wedge_\star \omega'$$

This gives a **differential calculus** on the deformed algebra of exterior forms $\Omega^\bullet_\star$.

Hodge star operator

- Key ingredient on a spacetime M is a metric.
  For an $n-$dimensional manifold M with metric g the Hodge $*$-operation is a linear map $* : \Omega^r(M) \to \Omega^{n-r}(M)$.

- In local coordinates an r-form is given by $\omega = \frac{1}{r!}\omega_{\mu_1 \dots \mu_r} \mathrm{d}x^{\mu_1} \wedge \dots \mathrm{d}x^{\mu_r}$ and the Hodge $*$-operator reads

$$*\omega = \frac{\sqrt{g}}{r!\,(n-r)!}\omega_{\mu_1 \dots \mu_r}\epsilon^{\mu_1 \dots \mu_r}{}_{\nu_{r+1} \dots \dots \nu_n}\mathrm{d}x^{\nu_{r+1}} \wedge \dots \mathrm{d}x^{\nu_n}$$

- the Hodge $*$-operator is A-linear: $*(\omega f) = *(\omega)f$, for any form ω and function f

- There is a one to one correspondence between metrics and Hodge star operators.
  Given a Hodge star, the metric is recovered via
  $dx^\mu \wedge *dx^\nu = g^{\mu\nu} Vol$

Twisted Hodge star operator

- We define the corresponding Hodge star operator on the $\star$-algebra of exterior forms $\Omega^\bullet_\star$.
- The deformed (twisted) Hodge operator $*^{\mathcal{F}}$ on $\Omega^\bullet_\star$ is required to map r-forms into $n-r$-forms, and to be right $A_\star$-linear

$$*^{\mathcal{F}}(\omega \star f) = *^{\mathcal{F}}(\omega) \star f$$

for any form ω and function f.

- **Canonical way** to deform A-linear maps to right $A_\star$-linear maps - the "quantization map" $\mathcal{D}$:

$$m \to \mathcal{D}(m) : \qquad \bar{\mathrm{f}}^\alpha_1 \triangleright \circ m \circ S(\bar{\mathrm{f}}^\alpha_2) \triangleright \circ \bar{\mathrm{f}}_\alpha \triangleright$$

G. Fiore, J. Math. Phys. 39 (1998), J. Phys. A 43 (2010);
P. Kulish, A. Mudrov, Lett. Math. Phys. 95 (2011);
P. Aschieri, A. Schenkel, Adv. Theor. Math. Phys. 18 (2014).

- The deformed or quantum Hodge $*$-operator:

$$\begin{array}{rcl} *^{\mathcal{F}} = \mathcal{D}(*) : \Omega^\bullet_\star & \longrightarrow & \Omega^\bullet_\star \\ \omega & \longmapsto & *^{\mathcal{F}}(\omega) = \bar{\mathrm{f}}^\alpha_1\Big(* \big(S(\bar{\mathrm{f}}^\alpha_2)\bar{\mathrm{f}}_\alpha(\omega)\big)\Big) \end{array}$$

- For any exterior form ω and function f we have the right $A_\star$-linearity property $*^{\mathcal{F}}(\omega \star f) = *^{\mathcal{F}}(\omega) \star f$.

Wave equation in curved spacetime

The Laplace-Beltrami operator is a generalization to curved spacetime of the D'Alembert operator.

- The wave equation in curved spacetime is governed by the Laplace-Beltrami operator $\Box = \delta \mathrm{d} + \mathrm{d}\delta$.
- In the case of even dimensional Lorenzian manifolds (like Minkowski spacetime) the adjoint of the exterior derivative is defined by $\delta = *\mathrm{d}*$
- For a scalar field φ we have (using local coordinates)

$$\Box_{LB}\varphi = *\mathrm{d} * \mathrm{d}\varphi = \frac{1}{\sqrt{g}}\partial_\nu \left[\sqrt{g} g^{\nu\mu}\partial_\mu\varphi\right]$$

Deformed Laplace-Beltrami operator

P. Aschieri, A. Borowiec, A.P., JHEP 152 (2017) [arXiv:1703.08726]
JCAP 04 (2021) [arXiv:2009.01051].

- Deformation of the Laplace-Beltrami operator for any twist:

$$\Box^{\mathcal{F}}_{LB}\varphi = *^{\mathcal{F}} d *^{\mathcal{F}} d\varphi$$

- κ-deformed wave equation by the Jordanian twist in curved background

$$\sqrt{g} g^{\mu\nu} \star (1 - \frac{i}{\kappa}\partial_0)^{n-2}\partial_\mu\partial_\nu\varphi + \partial_\nu(\sqrt{g} g^{\mu\nu}) \star (1 - \frac{i}{\kappa}\partial_0)^{n-1}\partial_\mu\varphi = 0$$

Dispersion relations in κ-FRWL case

P. Aschieri, A. Borowiec, A.P., JCAP 04 (2021) [arXiv:2009.01051].

Setting:

- a distant source that emits a gamma ray burst
- emitter and observer in first approximation do not have peculiar velocities and can be considered at rest with respect to the usual comoving coordinate system (t, x^i) of Friedman-Robertson-Walker-Lemaitre (FRWL) cosmology

Friedman-Robertson-Walker-Lemaitre (FRWL) metric:

$$g = -dt^2 + a^2(t)\sum_i (dx^i)^2$$

where $a(t)$ - scale factor

2-dim twisted κ-wave equation

$$\sqrt{g}g^{\mu\nu} \star \partial_\mu \partial_\nu \varphi + \partial_\nu(\sqrt{g}g^{\mu\nu}) \star (1 - \frac{i}{\kappa}\partial_0)\partial_\mu \varphi = 0$$

2-dim twisted κ-wave equation in FRWL background

$$-a \star \partial_0^2 \varphi - (\partial_0 a) \star \left(1 - \frac{i}{\kappa}\partial_0\right) \partial_0 \varphi + a^{-1} \star \partial_x^2 \varphi = 0$$

2-dim twisted κ-wave equation

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In the classical limit it reduces to:

$-a\partial_0^2\varphi - \dot{a}\partial_0\varphi + \frac{1}{a}\partial_i^2\varphi = 0$ where $\dot{a} = \partial_0 a(t)$

Classical version of equation

$$-a\partial_0^2\varphi - \dot{a}\partial_0\varphi + \frac{1}{a}\partial_i^2\varphi = 0$$

- separation of variables: $\varphi = \lambda\,(t)\, e^{-ikx}$
- $$a\ddot{\lambda} + \dot{\lambda}\dot{a} + k^2\lambda\frac{1}{a} = 0$$
- it corresponds (in conformal time) to harmonic oscillator type equation

$$(\partial_\eta^2 + k^2)\lambda = 0$$

Twisted wave equation

$$a \star \partial_0^2 \varphi + (\partial_0 a) \star \left(1 - \frac{i}{\kappa}\partial_0\right) \partial_0 \varphi - a^{-1} \star \partial_x^2 \varphi = 0$$

- In the noncommutative case in 2 dimensions we consider the solution of the form: $\varphi = \lambda(t) \star e^{-ikx} = \lambda(t)\, e^{-ikx}$

Twisted wave equation

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We simplify the equation as:

$$a \star \partial_0^2 \lambda + \partial_0 (a) \star \left(1 - \frac{i}{\kappa}\partial_0\right) \partial_0 \lambda + a^{-1} \star k^2 \lambda = 0$$

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$$a \star \partial_0^2 \lambda + \partial_0 (a) \star \left(1 - \frac{i}{\kappa}\partial_0\right) \partial_0 \lambda + a^{-1} \star k^2 \lambda = 0$$

Expand star-product in the first order of $\frac{1}{\kappa}$

$$a\partial_0^2\lambda + \partial_0 (a) \left(1 - \frac{i}{\kappa}\partial_0\right) \partial_0\lambda + a^{-1}k^2\lambda - \frac{i}{\kappa} t \left(\partial_0 a\, \partial_0^3\lambda + \partial_0^2 a\, \partial_0^2\lambda + k^2 \partial_0 a^{-1} \partial_0 \lambda\right) = 0$$

Conformal time - classical case strategy

- As in the classical case - change the coordinates into conformal time η, and $' = \partial_\eta$
- Introduce simplified notation $s = \ln a; s' = \frac{a'}{a}; \frac{a''}{a} = s'' + (s')^2;$

Conformal time - classical case strategy

- As in the classical case - change the coordinates into conformal time η, and $' = \partial_\eta$
- Introduce simplified notation $s = \ln a$; $s' = \frac{a'}{a}$; $\frac{a''}{a} = s'' + (s')^2$;
- Look for the solution of the type:

$$\lambda = \exp\left(i\omega\eta + \frac{i}{\kappa}F\right)$$

- Classical part (at 0-th order) remains:

$$\left(\omega^2 - k^2\right)\lambda = 0$$

- And equation on $F(\eta)$ becomes:

(using the zero-th order solution $\omega = k$),

$$F'' + 2ikF' = \frac{ikt(\eta)}{a^2}\left(2\left(s'\right)^3 - 2s's'' - 2k^2s' + ik\left(s'' - 3\left(s'\right)^2\right)\right) - \frac{ik}{a}s'\left(s' - ik\right) .$$

Group velocity for the wave

Starting from

$$\varphi_k(x,t) = \lambda(t) \star e^{-ikx} = \lambda(t) e^{-ikx} = \exp\left(ik\eta + \frac{i}{\kappa}F\right) e^{-ikx} = e^{i(f_k(t) - kx)}$$

we get:

$$f_k(t) = \left(k\eta + \frac{1}{\kappa}F\right)(t)$$

Group velocity expression

$$v_g = \frac{\partial x}{\partial t} = \frac{\partial}{\partial k}\frac{\partial f_k(t)}{\partial t}$$

$\Longrightarrow$ we need to compute $\dot{F} = \partial F/\partial t$.

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- can be obtained from the differential equation for F in the physical regime we are interested in:
  - cosmic time related to large scale structure formation,
  - and high frequency waves.
- There are three frequency parameters in the differential equation on F: $\omega = k$, t^{-1} and the Hubble parameter H;
- we have $\omega >> t^{-1}$ for the present cosmic time as well as the cosmic time of emission of the travelling γ-ray, typically at redshift below $z = 10$.
- Similarly $\omega >> H \sim t^{-1}$

In this regime equation for F simplifies to

$$2ikF' = -\frac{2ik^3 ts'}{a^2}$$

$$\dot{F} = -\frac{k^2 t \dot{a}}{a^3} \ .$$

- The group velocity, at the first order in the $\frac{1}{\kappa}$ deformation, results

$$v_g = \frac{\partial x}{\partial t} = \frac{\partial}{\partial k}\frac{\partial f_k(t)}{\partial t} = \frac{1}{a}+\frac{1}{\kappa}\frac{\partial \dot{F}}{\partial k} = \frac{1}{a}\Big(1-\frac{2}{\kappa}\frac{kt\dot{a}}{a^2}\Big) = \frac{1}{a}\Big(1-\frac{2}{\kappa}\frac{\omega t\dot{a}}{a^2}\Big)$$

- Taking into account the $\frac{1}{a}$ factor due to the comoving coordinates and inserting the flat spacetime speed of light c we see that κ-**spacetime noncommutativity in the presence of a FLRW metric** leads to a **velocity of photons** $v_{ph} = v_g a$ given by

$$v_{ph} = c(1 - \frac{2}{\kappa}\frac{\omega t\dot{a}}{a^2}) \ .$$

4-d case κ-FLRW case

$$\sqrt{g}g^{\mu\nu}\star(1-\frac{i}{\kappa}\partial_0)^2\partial_\nu\partial_\mu\varphi+\partial_\nu(\sqrt{g}g^{\mu\nu})\star(1-\frac{i}{\kappa}\partial_0)^3\partial_\mu\varphi-\frac{1}{6}(\sqrt{g}R)\star\varphi = 0\,.$$

gives the same result:

$$v_{ph} = c(1 - \frac{2}{\kappa}\frac{\omega t\dot{a}}{a^2})$$

- If we define the energy where the Planck scale (- Lorentz deformation) is manifested $E_P := |\kappa|\hbar$.
- The variation of the speed of light v_{ph} with respect to the usual one c (of photons in flat spacetime, or of low energetic photons) is then given by

$$|1 - v_{ph}/c| \sim \frac{E_{ph}}{E_P}\frac{2t\dot{a}}{a^2} \quad .$$

One can estimate the fractional variation of the speed of light by using:

$$\delta v/c \equiv |1 - v_{ph}/c| \sim 2(1+z)tH\, E_{ph}/E_P$$

Comments on the results

$$v_{ph} = c(1 - \frac{2}{\kappa}\frac{\omega t \dot{a}}{a^2}) = c(1 - \frac{E_{ph}}{E_P}\frac{2t\dot{a}}{a^2})$$

- The **combined effects of noncommutativity and gravity** affect the velocity of light by:
  - a term linearly dependent on the frequency ω,
  - the cosmic time t,
  - the Hubble parameter $H = \dot{a}/a$
  - and it is inversely proportional to the scale factor.

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  - and it is inversely proportional to the scale factor.
- In flat spacetime ($\dot{a} = 0$) as well as in commutative spacetime ($\kappa \to \infty$) there are no modified dispersion relations.
- This result offers an **explicit cosmological correction** to the usually considered models, which assume as the leading power for the correction to the light speed the expression $v_{ph} \;\sim\; c(1 - \frac{E_{ph}}{E_P})$.

Time lag Δt between the arrival of a low energetic and a high energetic photon emitted simultaneously during a gamma ray burst:

-

Considering only first order corrections - time delay Δt is

$$\Delta t = \frac{2E_{ph}}{E_{LV}} \int_{t_{em}}^{t_0} \frac{t\dot{a}}{a^3} dt = \frac{2E_{ph}}{E_{LV}} \int_0^z t\,(1+z')dz' \ .$$

- For the range of redshifts we are interested into (up to $z \sim 10$) we can use the analytic solution $a(t) = (1+z)^{-1} = (\frac{\Omega_m}{\Omega_\Lambda})^{1/3} \sinh^{2/3}(t/t_\Lambda)$, $t_\Lambda = \frac{2}{3H_0\sqrt{\Omega_\Lambda}}$ and obtain the time lag

$$\Delta t = 2\frac{E_{ph}}{E_{LV}} t_\Lambda \int_0^z \mathrm{arcsinh}\sqrt{\frac{\Omega_\Lambda}{\Omega_m}(1+z')^{-3}}\ (1+z')dz' \ .$$

Our model gives a time lag that is ~ 3 times the ones considered in the typical 'Lorentz invariance violation' literature.

Conclusions

- In the present work, as a first approximation, we have considered a commutative gravity background, hence noncommutativity affects only propagation of light.

- In a noncommutative theory of gravity consistently coupled to light, one could consider the backreaction effects of turning on noncommutativity also on the gravitational field.

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- Framework is valid for any **curved background** and any **noncommutative** spacetime (provided by twist).
- The result that the combined effects of noncommutativity and curvature produce modified dispersion relations is expected to be a **general feature** of wave equations in noncommutative curved spacetime.

Conclusions

- We used a **top-down approach that complements the bottom-up one of phenomenological models**
  - noncommutative deformation of the wave equation in curved background
  - modification of dispersion relations purely induced by noncommutativity and curvature of spacetime.

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Thank you for your attention!