Wykład 6 Mechanika Kwantowa

OSCYLATOR C.D.

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + i\sqrt{\frac{1}{2m\omega\hbar}}\,\hat{p} \qquad \text{op. anihilacji}$$

$$\hat{a}^+ = \qquad - \qquad\qquad \text{op. kreacji}$$

$$\hat{a}\hat{a}^+ = \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{1}{2m\omega\hbar}\hat{p}^2 - \frac{i}{2\hbar}[\hat{x},\hat{p}]$$

$$\hat{a}^+\hat{a} = \;-\|-\; \qquad -\|- \qquad + \frac{i}{2\hbar}[\hat{x},\hat{p}]$$

$$\boxed{[\hat{a},\hat{a}^+] = 1}$$

$$\{\hat{a},\hat{a}^+\} = \frac{m\omega}{\hbar}\hat{x}^2 + \frac{1}{m\omega\hbar}\hat{p}^2 =$$

$$= \frac{2}{\omega\hbar}\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right)$$

$$\rightarrow \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$$

$$\hat{a}^+\hat{a} = \hat{a}\hat{a}^+ - 1$$

$$= \frac{2}{\hbar\omega}\hat{H}_{osc}$$

$$\hat{H}_{osc} = \frac{\hbar\omega}{2}\{\hat{a},\hat{a}^+\} = \frac{\hbar\omega}{2}\big(\underbrace{\hat{a}\hat{a}^+}_{1+\hat{a}^+\hat{a}} + \hat{a}^+\hat{a}\big) = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

Szukamy $\hat{H}_{osc}|\alpha\rangle = E|\alpha\rangle$

Te stany są stanami własnymi $\hat{a}^+\hat{a}|\alpha\rangle = \alpha|$

Stany te unormujemy $\langle\alpha|\alpha\rangle = 1$.

Czy stan $|\beta\rangle = \hat{a}|\alpha\rangle$

jest stanem własnym $\hat{a}^+\hat{a}$? $\hat{a}^+\hat{a}|\beta\rangle \overset{?}{=} \beta|\beta\rangle$

$$\hat{a}^+\hat{a}|\beta\rangle = \hat{a}^+\hat{a}\,\hat{a}|\alpha\rangle = (\hat{a}\hat{a}^+ - 1)\hat{a}|\alpha\rangle =$$

$$= \hat{a}(\hat{a}^+\hat{a} - 1)|\alpha\rangle = (\alpha - 1)\hat{a}|\alpha\rangle =$$

$$= (\alpha - 1)|\beta\rangle$$

$$\hat{a}|\alpha\rangle = m_-|\alpha-1\rangle$$
$$\hat{a}^\dagger|\alpha\rangle = m_+|\alpha+1\rangle$$

← Obliczyć normalizacja

$$\langle\beta|\beta\rangle = \underbrace{\langle\alpha|\hat{a}^\dagger}_{\langle\beta|}\underbrace{\hat{a}|\alpha\rangle}_{|\beta\rangle} = \alpha\underbrace{\langle\alpha|\alpha\rangle}_{=1} = \alpha$$

$$\hookrightarrow m_-^2\underbrace{\langle\alpha-1|\alpha-1\rangle}_{=1} = m_-^2$$

$$\boxed{m_- = \sqrt{\alpha}}$$

$$|\gamma\rangle = \hat{a}^\dagger|\alpha\rangle$$

$$\langle\gamma|\gamma\rangle = \langle\alpha|\hat{a}\hat{a}^\dagger|\alpha\rangle = \langle\alpha|1+\hat{a}^\dagger\hat{a}|\alpha\rangle = 1+\alpha = m_+^2$$

$$\boxed{m_+ = \sqrt{1+\alpha}}$$

$$\rightarrow \boxed{\begin{aligned}\hat{a}|\alpha\rangle &= \sqrt{\alpha}\,|\alpha-1\rangle\\ \hat{a}^\dagger|\alpha\rangle &= \sqrt{1+\alpha}\,|\alpha+1\rangle\end{aligned}}$$

$$H_{osc}|\alpha\rangle = \hbar\omega\left(\alpha+\frac{1}{2}\right)|\alpha\rangle \qquad \alpha\in\mathbb{R}$$

MUSI ISTNIEĆ STAN PODSTAWOWY. $\rightsquigarrow \alpha = n \in \mathbb{N}$

$$\hat{a}|0\rangle = \sqrt{0}\,|-1\rangle = 0$$
$$\equiv$$

$$\boxed{\begin{aligned}\hat{a}|n\rangle &= \sqrt{n}\,|n-1\rangle\\ \hat{a}^\dagger|n\rangle &= \sqrt{1+n}\,|n+1\rangle\end{aligned}} \qquad E_{osc}^{(n)} = \hbar\omega\left(n+\frac{1}{2}\right)$$

Konstrukcja stanów wł. H_{osc} :

$$|n\rangle = \frac{1}{\quad}\left(\hat{a}^\dagger\right)^n|0\rangle$$

$\sqrt{n!}$

Reprezentacja macierzowa:

$\langle m|$ $\qquad \hat{a}|n\rangle = \sum_{m=0}^{\infty} |m\rangle a_{mn} \qquad \langle m|\hat{a}|n\rangle = a_{mn}$

$$\hat{a}^+|n\rangle = \sum_{m=0}^{\infty} |m\rangle a^+_{mn}$$

$$a = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ \cdots & & & & \ddots \end{bmatrix} \qquad a^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & \cdots \\ \cdots & & & & \end{bmatrix}$$

$$|n\rangle \Rightarrow \begin{array}{cl} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} & \begin{array}{l} \leftarrow n=0 \\ \\ \leftarrow n \\ \\ \end{array} \end{array} \qquad |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$a|0\rangle = \begin{bmatrix} 0 & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a^+|0\rangle = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ \cdots & & & & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \approx |1\rangle$$

DYGRESJA: $\quad [\hat{A}, \hat{B}] = c\,\hat{\mathbb{1}}$

Dla takich op. nie istnieje sk. wym. reprezentacja macierzowa.

(wymiar n)

$$Tr(\underbrace{AB}) - Tr(\underbrace{BA}) = 0 \qquad Tr(c\,\mathbb{1}) = cn$$

$n \to \infty$ lewa strona jest ciągle $= 0$ a prawa $\to \infty$

$[\hat{a}, \hat{a}^+] = 1$

$[\hat{a}, \hat{a}^\dagger] = 1$

$$\langle n|[\hat{a}, \hat{a}^\dagger]|n\rangle = \underbrace{\langle n|\hat{a}\hat{a}^\dagger|n\rangle}_{n+1} - \underbrace{\langle n|\hat{a}^\dagger\hat{a}|n\rangle}_{n} = 1$$

Ślad: $\quad L = \sum_{n=0}^{\infty} 1 \qquad P = \sum_{n=0}^{\infty} 1$

Inny spsób: $(1+2+3+\dots) - (0+1+2+3\dots) \stackrel{?}{=} 0$

Własności stanów $|n\rangle$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}) \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})$$

$$\langle n|\hat{x}|n\rangle = 0 \qquad \langle n|\hat{p}|n\rangle = 0$$

Średnie odchył kwadratowe

$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|(\underbrace{\hat{a}^\dagger\hat{a}^\dagger}_{=0} + \underbrace{\hat{a}\hat{a}}_{=0} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)|n\rangle = \frac{\hbar}{2m\omega}(2n\cdot$$

$$\langle n|\hat{p}^2|n\rangle = -\frac{m\omega\hbar}{2}\langle n| \;-\,||\,-\; -\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger|n\rangle = \frac{m\omega\hbar}{2}(2n+$$

$$\langle n|\hat{x}^2|n\rangle\langle n|\hat{p}^2|n\rangle = \frac{\hbar^2}{4}(2n+1)^2 \geqslant \frac{\hbar^2}{4}$$

Dla $n=0$ Z.N. jest wysycona

Reprezentacja położeniowa:

$$\varphi_n(x) = \langle x|n\rangle$$

Stan podstawowy $|0\rangle \qquad \hat{a}|0\rangle = 0$

$$\hat{a}\varphi_0(x) = 0 \to \left(\sqrt{\frac{m\omega}{2\hbar}}\,x + i\sqrt{\frac{1}{2m\omega\hbar}}\left(-i\hbar\frac{\partial}{\partial x}\right)\right)\varphi_0(x$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_0(x)=0$$

$$\psi_0(x)=\mathcal{N}e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} \qquad \mathcal{N}^2=\sqrt{\frac{m\omega}{\pi\hbar}}$$

Nowe zmienne: $\boxed{\xi=\sqrt{\frac{m\omega}{\hbar}}\,x}$ $\quad \psi_0(\xi)=\mathcal{N}e^{-\frac{1}{2}\xi^2}$

$$\hat{a}=\sqrt{\frac{1}{2}}\left(\xi+\frac{d}{d\xi}\right) \qquad \hat{a}^+=\sqrt{\frac{1}{2}}\left(\xi-\frac{d}{d\xi}\right)$$

Musimy policzyć $\psi_n(\xi)$ korzystając z faktu, że

$$\psi_n(\xi)=\frac{1}{\sqrt{n!}}(\hat{a}^+)^n\psi_0(\xi)$$

Własność:

$$\sqrt{\frac{1}{2}}\left(\xi-\frac{d}{d\xi}\right)f(\xi)=-\sqrt{\frac{1}{2}}\,e^{\frac{1}{2}\xi^2}\frac{d}{d\xi}\left(e^{-\frac{1}{2}\xi^2}f(\xi)\right)$$

$$=-\sqrt{\frac{1}{2}}\,e^{\frac{1}{2}\xi^2}\left(e^{-1/2\,\xi^2}(-\xi)+e^{-1/2\,\xi^2}\frac{d}{d\xi}\right)f(\xi)$$

Można pokazać:

$$\left[\sqrt{\frac{1}{2}}\left(\xi-\frac{d}{d\xi}\right)\right]^n f(\xi)=(-)^n\sqrt{\frac{1}{2^n}}\,e^{\frac{1}{2}\xi^2}\frac{d^n}{d\xi^n}\left[e^{-1/2\,\xi^2}f(\xi)\right]$$

Weźmy za $f(\xi)=\psi_0(\xi)=\mathcal{N}e^{-1/2\,\xi^2}$

$$\psi_n(\xi)=\frac{1}{\sqrt{n!}}(\hat{a}^+)^n\psi_0(\xi)=(-)^n\frac{1}{\sqrt{n!}}\sqrt{\frac{1}{2^n}}\,e^{1/2\,\xi^2}\frac{d^n}{d\xi^n}\left(\mathcal{N}e^{-\xi^2}\right)$$

$$=\mathcal{N}e^{-1/2\,\xi^2}\frac{1}{\quad}(-)^n e^{\xi^2}\frac{d^n}{\quad}e^{-\xi^2}$$

$$\sqrt{n!\,2^n} \qquad \underbrace{\qquad\qquad d\xi^n}_{H_n(\xi)}$$

$\omega_n = \frac{E_n}{\hbar}$ Dowolny stan $|\psi(x,t)\rangle = \sum_n c_n e^{-i\frac{E_n t}{\hbar}} |n\rangle$

$$\langle\psi|\hat{x}|\psi\rangle = \sum_{m,n} c_m^* c_n e^{-i(n-m)t} \langle m|\hat{x}|n\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \sum_{n=0}^{\infty} \left(\sqrt{n+1}\, c_{n+1}^* c_n e^{i\omega t} + \sqrt{n}\, c_{n-1}^* c_n e^{-i\omega t} \right)$$

$$= \sum_n X_n \cos(\omega t + \varphi_n)$$

$$e^{i\varphi_n} X_n = \sqrt{\frac{\hbar}{2m\omega}} \left(c_n^* c_{n-1} \right)$$