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Ruch w polu magnetycznym – poziomy Landaua

Siła Lorentza: $\vec{F} = q\vec{E} + \frac{q}{c}(\vec{v} \times \vec{B})$

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}} \qquad \vec{E} = -\vec{\nabla} V$$

$$L = \frac{1}{2} m v^2 - qV + \frac{q}{c} \vec{A} \cdot \vec{v}$$

Hamiltonian: $\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + \frac{q}{c}\vec{A}$

$$\vec{v} = \frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)$$

$$H = p \cdot v - L$$

$$\boxed{H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + qV}$$

Potencjał $\vec{A}$ nie jest jednoznaczny.
Musimy wybrać cechowanie.
Ruch w stałym polu mgn $\parallel \hat{z}$: $B_z = B$

$$\vec{A} = B \begin{bmatrix} -y \\ 0 \\ 0 \end{bmatrix} \qquad B_z = \underbrace{\partial_x A_y}_{=0} - \partial_y A_x = B$$

$$B_x = \partial_y A_z - \partial_z A_y = 0$$

$$\hat{H} = \frac{1}{2m}\left(\hat{\vec{p}} - \frac{q}{c}\hat{\vec{A}}\right)^2 = \frac{1}{2m}\left(\left(\hat{p}_x + \frac{qB}{c} y\right)^2 + \hat{p}_y^2 + \hat{p}_z^2\right) =$$

$$= \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_z^2\right) + \frac{1}{2m}\hat{p}_y^2 + \underbrace{\frac{qB}{mc}}_{\tilde{\omega} = 2\omega} \hat{p}_x y + \underbrace{\left(\frac{qB}{mc}\right)^2}_{\tilde{\omega}^2 = (2\omega)^2} \frac{1}{2} m y^2$$

$$= \frac{1}{2m}\left(\cancel{\hat{p}_x^2} + \hat{p}_z^2\right) + \frac{1}{2m}\hat{p}_y^2 + \frac{1}{2}m\tilde{\omega}^2\left(y^2 + 2\frac{\hat{p}_x}{m\tilde{\omega}}y + \left(\frac{1}{m\tilde{\omega}}\right)^2\hat{p}_x^2\right) -$$

$$- \frac{1}{2}m\tilde{\omega}^2\cancel{\left(\frac{1}{m\tilde{\omega}}\right)^2}\hat{p}_x^2$$

$$\hat{H} = \frac{1}{2m}\hat{p}_z^2 + \frac{1}{2m}\hat{p}_y^2 + \frac{1}{2}m\tilde{\omega}^2\left(y^2 + 2\frac{\hat{p}_x y}{m\tilde{\omega}} + \frac{1}{m^2\tilde{\omega}^2}\hat{p}_x^2\right)$$

$$\hat{H}\psi(x,y,z) = E\psi(x,y,z) \qquad \psi = f(y)\,e^{\frac{i p_x \cdot x}{\hbar}}\,e^{\frac{i p_z \cdot z}{\hbar}}$$

Działając $\hat{H}\psi$ $\quad \hat{p}_x, \hat{p}_z \rightarrow p_x, p_z \leftarrow$ wartości własne

$$\hat{H} = \frac{1}{2m}p_z^2 + \underbrace{\frac{1}{2m}\hat{p}_{y'}^2 + \frac{1}{2}m\tilde{\omega}^2 y'^2}$$

$$y' = y + \frac{p_x}{m\tilde{\omega}}$$

Równ. Schr. $\quad \hat{H}f(y') = E f(y')$

$\downarrow$ oscylator

$$E_n = \frac{1}{2m}p_z^2 + \underbrace{\hbar\tilde{\omega}\left(n + \frac{1}{2}\right)}_{\hbar\omega(2n+1)}$$

$$\boxed{\omega = \frac{qB}{2mc}}$$

ruch swobodny wzdłuż z

oscylator

Degeneracja?

Nieskończona degeneracja ze względu na p_x

Inny wybór cechowania: $\vec{A} = \frac{1}{2}B\begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$

$$\hat{H} = \frac{1}{2m}\left[\left(\hat{p}_x + \frac{qB}{2c}y\right)^2 + \left(\hat{p}_y - \frac{qB}{2c}x\right)^2\right] + \frac{\hat{p}_z^2}{2m}$$

$$= \frac{1}{2}\hbar\tilde{\omega}\left[\frac{\left(\hat{p}_x + \frac{1}{2}m\tilde{\omega}y\right)^2}{m\hbar\tilde{\omega}} + \frac{\left(\hat{p}_y - \frac{1}{2}m\tilde{\omega}x\right)^2}{m\hbar\tilde{\omega}}\right] + \frac{\hat{p}_z^2}{2m}$$

$$\tilde{\omega} = \frac{qB}{mc}$$

Definicja:
$$\hat{\pi}_x = \frac{1}{\sqrt{m\hbar\tilde{\omega}}}\left(\hat{p}_x + \frac{1}{2}m\tilde{\omega}y\right)$$
$$\hat{\pi}_y = \frac{1}{\sqrt{m\hbar\tilde{\omega}}}\left(\hat{p}_y - \frac{1}{2}m\tilde{\omega}x\right)$$

$$\hat{H} = \frac{1}{2}\hbar\tilde{\omega}\left\{\hat{\pi}_x^2 + \hat{\pi}_y^2\right\} + \frac{\hat{p}_z^2}{2m}$$

Komutator:

$$\left[\hat{\pi}_x, \hat{\pi}_y\right] = \frac{1}{\cancel{m}\hbar\cancel{\tilde{\omega}}}\left\{-\frac{1}{2}\cancel{m\tilde{\omega}}\left[\hat{p}_x, x\right] + \frac{1}{2}\cancel{m\tilde{\omega}}\left[y, \hat{p}_y\right]\right\} = i$$

Możemy zgadnąć op. kreacji i anihilacji

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hat{\pi}_x + i\hat{\pi}_y\right) \qquad \hat{a}^+ = \frac{1}{\sqrt{2}}\left(\hat{\pi}_x - i\hat{\pi}_y\right)$$

$$\left[\hat{a}, \hat{a}^+\right] = \frac{1}{2}i\left\{-\left[\hat{\pi}_x, \hat{\pi}_y\right] + \left[\hat{\pi}_y, \hat{\pi}_x\right]\right\} = 1$$

$$\frac{1}{2}\left(\hat{\pi}_x^2 + \hat{\pi}_y^2\right) = \hat{a}^+\hat{a} + \frac{1}{2}$$

$$\hat{H} = \hbar\tilde{\omega}\left[\hat{a}^+\hat{a} + \frac{1}{2}\right] + \frac{\hat{p}_z^2}{2m}$$

$$E_n = \hbar\tilde{\omega}\left(n + \frac{1}{2}\right) + \frac{p_z^2}{2m}$$

Degeneracja $\longrightarrow$ ćwiczenia.

OBRAZY.

R. Schr.
$$i\hbar \frac{d}{dt} |\alpha_S(t)\rangle = \hat{H}_S |\alpha_S(t)\rangle$$

$$\underline{-i\hbar} \frac{d}{dt} \langle \alpha_S(t)| = \langle \alpha_S(t)| \hat{H}_S$$

Rozw.
$$|\alpha_S(t)\rangle = e^{-i\frac{\hat{H}_S t}{\hbar}} |\alpha_S(0)\rangle$$
ewolucja czasow od $t_0 = 0$

$$\langle \alpha_S(t)| = \langle \alpha_S(0)| e^{+i\frac{\hat{H}_S t}{\hbar}}$$

Op. ewolucji
$$\hat{U}(t) = e^{-i\frac{\hat{H}_S t}{\hbar}}$$

Mamy jakiś op $\hat{O}_S(t)$

jawna zal. od czasu.

$$\frac{d}{dt} \langle \alpha_S(t)| O_S(t) |\beta_S(t)\rangle = \langle \alpha_S(t)| \frac{\partial}{\partial t} O_S(t) |\beta_S(t)\rangle$$
$$+ \frac{1}{i\hbar} \langle \alpha_S(t)| [\hat{O}_S(t), \hat{H}_S] |\beta_S(t)\rangle$$

OBRAZ HEISENBERGA

$$\frac{d}{dt} \langle \alpha_S(0)| e^{i\frac{\hat{H}_S t}{\hbar}} \hat{O}_S(t) e^{-i\frac{\hat{H}_S t}{\hbar}} |\beta_S(0)\rangle$$
$$= \langle \alpha_S(0)| e^{i\frac{\hat{H}_S t}{\hbar}} \frac{\partial}{\partial t} \hat{O}(t) e^{-i\frac{\hat{H}_S t}{\hbar}} |\beta_S(0)\rangle$$
$$+ \frac{1}{i\hbar} \underbrace{\langle \alpha_S(0)|}_{\langle \alpha_H|} \Big[\underbrace{e^{i\frac{\hat{H}_S t}{\hbar}} \hat{O}(t) e^{-i\frac{\hat{H}_S t}{\hbar}}}_{\hat{O}_H(t)}, \underbrace{\hat{H}_S}_{= \hat{H}_H \to \hat{H}} \Big] |\beta_S(0)\rangle$$

ewolucja w czasie

OBR. SCHR. - stany ewoluują
- op. nie (za wyjątkiem jawnej zal. od czasu)

OBR. HEIS. - stany nie ewoluują
- operatory ew. w czasie

$$\langle\alpha_H|\frac{d}{dt}\hat{O}_H(t)|\beta_H\rangle = \underline{\langle\alpha_H|\frac{\partial}{\partial t}\hat{O}_H(t)|\beta_H\rangle} + \frac{1}{i\hbar}\langle\alpha_H|[\hat{O}_H(t),\hat{H}]|\beta_H\rangle$$

$$\frac{\partial}{\partial t}\hat{O}_H(t) = e^{\frac{i\hat{H}t}{\hbar}}\frac{\partial}{\partial t}O_S(t)e^{-\frac{i\hat{H}t}{\hbar}}$$

→ Równanie operatorowe

$$\frac{d}{dt}\hat{O}_H(t) = \frac{\partial}{\partial t}\hat{O}_H(t) + \frac{1}{i\hbar}[\hat{O}_H(t),\hat{H}]$$

OBRAZ ODDZIAŁYWANIA (TOMONAGI)

$$\hat{H} = \hat{H}_0 + \hat{H}'$$

stany $|\alpha_I(t)\rangle = e^{\frac{i\hat{H}_0 t}{\hbar}}|\alpha_S(t)\rangle$

$$\hat{O}_I(t) = e^{\frac{i\hat{H}_0 t}{\hbar}}\hat{O}_S e^{-\frac{i\hat{H}_0 t}{\hbar}}$$

$$i\hbar\frac{d}{dt}|\alpha_I(t)\rangle = -H_0|\alpha_I(t)\rangle + \underbrace{e^{\frac{iH_0 t}{\hbar}}\hat{H}e^{-\frac{iH_0 t}{\hbar}}}_{\hat{H}_I}\underbrace{e^{+\frac{iH_0 t}{\hbar}}|\alpha_S(t)\rangle}_{|\alpha_I(t)\rangle}$$

$$H_I = e^{\frac{iH_0t}{\hbar}} (H_0 + H') e^{-\frac{iH_0t}{\hbar}} =$$

$$= H_0 + H_I'$$

$$i\hbar \frac{d}{dt} |\alpha_I(t)\rangle = H_I' |\alpha_I(t)\rangle$$

Rów. na operatory: $\frac{dO_I}{dt} = \frac{\partial O_I}{\partial t} + \frac{1}{i\hbar}[\hat{O}_I, \hat{f}$

ZWIĄZEK Z FIZYKĄ KL.

Zas. wariacyjna

$$S = \int L(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n)\, dt$$

$$\longrightarrow \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

definiujemy pędy uogólnione $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$$H(q_1 \dots q_n, p_1 \dots p_n) = \sum_{i=1}^{n} p_i \cdot \dot{q}_i(p) - L(q_1 \dots q_n, \dot{q}_1(p) \dots \dot{q}_n(p))$$

R. Hamiltona-Jacobiego

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Zależność od czasu funkcji $F(q_1 \dots q_n, p_1 \dots p_n, t)$

$$\frac{d}{dt} F(\dots) = \frac{\partial F}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial p_i} \dot{p}_i \right)$$

$$= \frac{\partial F}{\partial t} + \underbrace{\sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right)}_{\text{nawias Poissona } \{F, H\}}$$

$$\frac{d}{dt} F = \frac{\partial F}{\partial t} + \{F, H\}$$

Analogiczne równ. do równ. Heisenberga

$$\{F, H\} \rightarrow \frac{1}{i\hbar} [\hat{F}, \hat{H}]$$

$$\{q_i, p_j\} = \delta_{ij} \qquad [\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}$$

Kwantyzacja układu klasycznego polega na zastąpieniu nawiasów Poissona komutatorami. $\left(\frac{1}{i\hbar}\right)$