# Mechanika Kwantowa, wykład 10, 11.5.2021

PRZYBLIŻENIE PÓŁKLASYCZNE (SEMIKLASYCZNE, QUASIKLASYCZNE, WKB)

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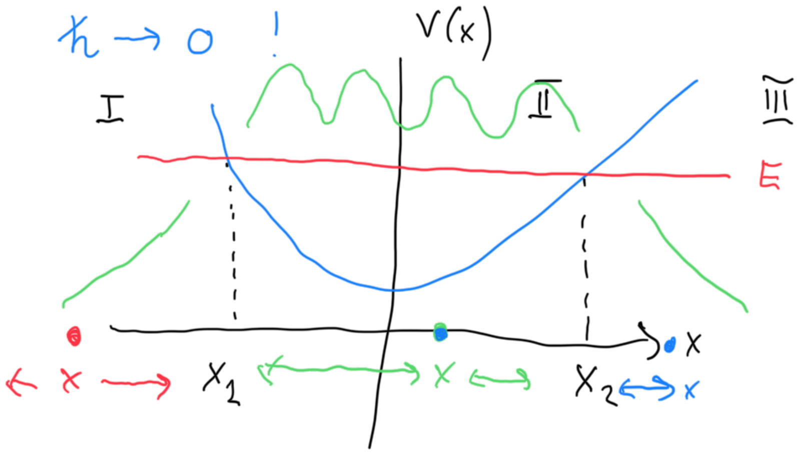
$\hbar \to 0$ !




$x_{1,2}$ – klasyczne punkty zwrotu

R. Schr.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\vec{\nabla}^2 \psi + V\psi$$

$$\psi = A\, e^{-i(Et - S(\vec{r}))/\hbar}$$

Równanie na fazę S:

$$\boxed{E = \frac{1}{2m}\left(\vec{\nabla} S\right)^2 - i\frac{\hbar}{2m}\vec{\nabla}^2 S + V}$$

$$\vec{\nabla}^2 e^{iS/\hbar} = \vec{\nabla}\left(\frac{i}{\hbar}\vec{\nabla} S\, e^{iS/\hbar}\right) =$$

$$= \left(\frac{i}{\hbar}\vec{\nabla}^2 \cdot S - \frac{1}{\hbar^2}\left(\vec{\nabla}\cdot S\right)^2\right) e^{iS/\hbar}$$

W jednym wymiarze: $' = \frac{d}{dx}$

$$\boxed{(S')^2 - 2m(E - V) - i\hbar S'' = 0}$$

$S = S_0 + \hbar S_1 + \dots$

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$$S_0'^2 + 2\hbar S_0' S_1' - 2m(E-V) - i\hbar S_0'' = 0$$

$\hbar^0$: $$S_0'^2 = 2m(E - V(x))$$
$$S_0' = \pm\sqrt{2m(E-V(x))}$$

$$\boxed{S_0(x) = \pm \int^x dx' \sqrt{2m(E-V(x'))} + \text{const.}}$$

$\hbar^1$: $$S_0'' = -2i S_0' S_1'$$

$$\frac{S_0''}{S_0'} = -2i S_1' \qquad \frac{d}{dx} \ln S_0'(x)$$

$$S_1 = \frac{1}{2} i \ln S_0'(x) = \frac{i}{2} \ln p(x) \quad (+\text{const.})$$

Dwa rozw. na $\psi$

$$\psi_\pm(x) = A e^{\pm i \int^x dx' p(x')/\hbar} e^{-\ln\sqrt{p(x)}}$$

$$= \frac{A}{\sqrt{p(x)}} e^{\pm i \int^x dx' p(x')/\hbar}$$

$$k(x) = \frac{1}{\hbar}\sqrt{2m(E-V(x))} \qquad \varkappa = \frac{1}{\hbar}\sqrt{2m(V(x)-E)} \leftarrow$$

II          I, III

I: $$\psi_I(x) = \frac{A}{2\sqrt{\varkappa(x)}} e^{-\int_x^{x_1} dx' \varkappa(x')}$$

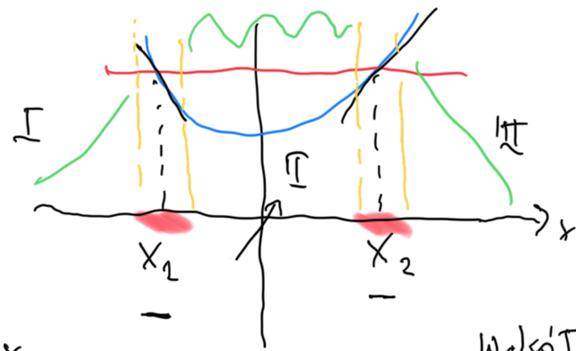
II: $$\psi_{II}^{(1)}(x) = \frac{C_1}{\sqrt{k(x)}} e^{i\int_{x_1}^x dx' k(x')} + \frac{C_2}{\sqrt{k(x)}} e^{-i\int_{x_1}^x dx' k(x')}$$

$$\to \quad \psi_{II}^{(2)}(x) = \frac{D_1}{\sqrt{k(x)}} e^{i\int_x^{x_2} dx' k(x')} + \frac{D_2}{\sqrt{k(x)}} e^{-i\int_x^{x_2} dx' k(x')}$$

$$- \qquad e^{-\int^x dx' \varkappa(x')}$$

→ $\underline{III}$ $\quad \psi_{III}(x) = \frac{B}{2\sqrt{\varkappa(x)}}\, e^{\cdots_{x_2}}$

$k(x_1) = \varkappa(x_1) = 0 \qquad k(x_2) = \varkappa(x_2) = 0$




W obszarze $x = x_1 \pm \varrho$
$x = x_2 \pm \varrho$
przybliżymy $V(x)$
przez prostą.

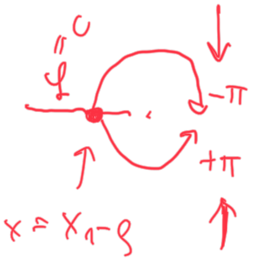
Wokół $x_1$ $\qquad$ Wokół $x_2$

$V(x) = E - F_1\cdot(x - x_1) + \ldots \qquad V(x) = E + F_2\cdot(x - x_2) + \ldots$

$F_{1,2} > 0$


$\varkappa(x) = \frac{1}{\hbar}\sqrt{2mF_1}\,(\underbrace{x_1 - x}_{\varrho^{1/2}})^{1/2} \qquad k(x) = \frac{1}{\hbar}\sqrt{2mF_1}\,(\underbrace{x - x_1}_{\varrho^{1/2}})^{1/2}$

$\int_x^{x_1} dx'\varkappa(x') = \frac{\sqrt{2mF_1}}{\hbar}\,\underline{\frac{2}{3}(x_1 - x)^{3/2}} \qquad \int_{x_1}^{x} dx' k(x') = \frac{\sqrt{2mF_1}}{\hbar}\,\frac{2}{3}(x - x_1)^{3/2}$




$(x_1 - x) = \varrho\, \underline{e^{i\varphi}} \longrightarrow e^{\mp i\pi}\varrho = e^{\mp i\pi}(x - x_1)$

$(x_1 - x)^{3/2} = \underbrace{\varrho^{3/2} e^{i\frac{3\varphi}{2}}}_{\varphi = 0} \longrightarrow e^{\mp i\frac{3\pi}{2}}\varrho^{3/2} = \pm i\,\varrho^{3/2}$

$= \pm i\,(x - x_1)^{3/2}$

$-\int_x^{x_1} dx'\varkappa(x') \longrightarrow \mp i \int_{x_1}^{x} dx' k(x')$

górny półokrąg $\quad \sqrt{\varkappa(x)} = \sqrt[4]{2mF_1}\,(x_1 - x)^{1/4} \longrightarrow$

$$\to \sqrt[4]{2mF_1}\,\underbrace{(x-x_1)^{1/4}}_{\sqrt{k(x)}}\, e^{-i\pi/4}$$

$$\sqrt{\kappa(x)} \to \sqrt{k(x)}\, e^{-i\pi/4}$$

$$\frac{A}{2\sqrt{\varkappa(x)}}\, e^{-\int_x^{x_1} dx'\varkappa(x')} \to \underbrace{\frac{A e^{+i\pi/4}}{2\sqrt{k(x)}}}_{C_2/\sqrt{k(x)}}\, e^{-i\int_{x_1}^{x} dx' k(x')}$$

$$C_2 = \frac{A e^{i\pi/4}}{2}$$

$$\frac{A e^{-i\pi/4}}{2\sqrt{k(x)}}\, e^{+i\int_{x_1}^{x} dx' k(x')}$$

Przejście dołem

$$C_1 = \frac{A e^{-i\pi/4}}{2}$$

$$\psi_{II}^{(1)}(x) = \frac{A}{2\sqrt{k(x)}}\left(e^{+i\int_{x_1}^{x} dx' k(x') - i\frac{\pi}{4}} + e^{-i\int_{x_1}^{x} dx' k(x') + i\frac{\pi}{4}}\right)$$

$$\psi_{II}^{(1)}(x) = \frac{A}{\sqrt{k(x)}} \cos\left(\int_{x_1}^{x} dx' k(x') - \frac{\pi}{4}\right)$$

Przejście od $x > x_2$ do $x < x_2$ daje

$$\psi_{II}^{(2)}(x) = \frac{B}{\sqrt{k(x)}} \cos\left(\int_{x}^{x_2} dx' k(x') - \frac{\pi}{4}\right)$$

Musi zajść $\psi_{II}^{(1)}(x) = \psi_{II}^{(2)}(x)$

$$\cos\left(\int_{x_1}^{x} dx' k(x') - \frac{\pi}{4}\right) = \cos\left(\int_{x_1}^{x_2} dx' k(x') - \underline{\int_{x}^{x_2} dx' k(x')} + \frac{\pi}{4} - \frac{\pi}{2}\right)$$

$$= \cos\left(\int_{x}^{x_2} dx' k(x') - \frac{\pi}{4} - \left[\int_{x_1}^{x_2} dx' k(x') - \frac{\pi}{2}\right]\right) \approx \psi_{II}^{(2)}(x$$

$$\cos(\varphi - \pi n) = (-)^n \cos\varphi \qquad \underbrace{\phantom{xx}}_{\pi n}$$

$$\boxed{B = (-)^n A}$$

WARUNEK KWANTYZACJI:

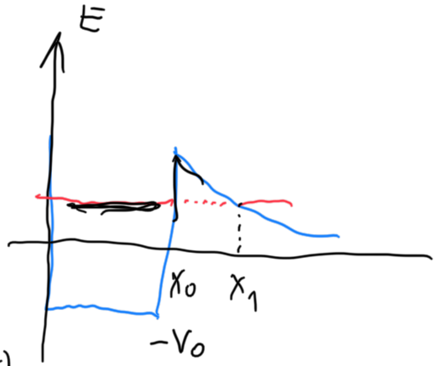
$$\boxed{\int_{x_1}^{x_2} dx'\, p(x') = \hbar\pi\left(n + \frac{1}{2}\right)}$$

WARUNEK BOHRA – SOMMERFELDA

$$\oint dx\, p(x) = 2\pi\hbar\left(n + \frac{1}{2}\right)$$

$$p(x) = \sqrt{2m\left(E - V(x)\right)}$$

TUNELOWANIE




$$\psi(x_1) = \psi(x_0)\underbrace{e^{-\int_{x_0}^{x_1} dx\, \varkappa(x)}}$$

T = współczynnik transmisji

$$= \psi(x_0)\, e^{-\gamma/2}$$

Wewnątrz studni $v = \frac{p}{m} = \sqrt{\frac{2(E+V_0)}{m}}$

Częstość uderzeń w barierę.

$$\nu = \frac{v}{2x_0}$$

Za każdym uderzeniem w $x_0$ prawd. przejścia

$P = e^{-\gamma}$

Tempo rozpadu (decay rate)

$$R = P \cdot v' = \sqrt{\frac{E+V_0}{2m}}\,\frac{1}{x_0}\,e^{-\gamma}$$