Mechanika Kwantowa dla doktorantów

zestaw 21 na dzień 20.04.2017 godz. 8:15

1. Finish the problem from the last set:

   Consider scattering on the repelling potential ($\gamma > 0$)

   $$\frac{2m}{\hbar^2} V(r) = \gamma \delta(r - R).$$

   Write Schrödinger equation and then for $l = 0$ solve it on the left and on the right of the potential and glue the solutions appropriately. Then calculate δ_0. Consider the case when γ is very large. Next, show that for any γ but small kR function $\cot \delta_0$ exhibits resonant behaviour in scattering energy E (namely $\cot \delta_0 = -c(E - E_r)$, which tends to zero for $E = E_r$). Find position of these resonances

   Prove that $\cot \delta_0$ goes through zero from above for increasing k. Find widths of these respnances.

2. Consider $f^{(1)}(\theta)$ in Born approximation for the Yukawa potential (set 18, problem3):

   $$V(r) = V_0 \frac{e^{-\mu r}}{\mu r}.$$

   Assuming that phase shifts are small $|\delta_l| \ll 1$, derive formula for δ_l expressed through Legendre'a functions $Q_l(\zeta)$ defined as:

   $$Q_l(\zeta) = \frac{1}{2} \int_{-1}^{1} d\zeta' \frac{P_l(\zeta')}{\zeta - \zeta'}.$$

   Note that the Legendre equation has two linearly independent solutions P_l and Q_l, the latter one being logarithmically divergent. Prove, by substituting to the Legendre equation, that Q_l given by the above integral is the solution.

   Derive asymptotic expansion for $|\zeta| > 1$

   $$Q_l(\zeta) = \frac{l!}{1 \cdot 3 \ldots (2l+1)} \times \left\{ \frac{1}{\zeta^{l+1}} + \frac{(l+1)(l+2)}{2 \cdot (2l+3)} \frac{1}{\zeta^{l+3}} + \frac{(l+1)(l+2)(l+3)(l+4)}{2 \cdot 4 \cdot (2l+3)(2l+5)} \frac{1}{\zeta^{l+5}} + \ldots \right\} \quad (1)$$

   Using expansion (1) show the following:

   (a) δ_l is positive (negative) for $V_0 < 0$ ($V_0 > 0$);

   (b) for de Broglie wave length larger than the potential range the phase shifts take the following form:

   $$\delta_l = \text{const} \times k^{2l+1}.$$

(c) find formula for const.

3. The Lippmann-Schwinger formalism can be also applied to a *one-dimensional* transmission-reflection problem with a finite range potential, $V(x) \neq 0$ for $-a < x < a$ only.

   - Suppose we have an incident wave coming from the left $\langle x | \phi \rangle = e^{ikx}/\sqrt{2\pi}$. How must we handle the singular $1/(E - H_0)$ operator if we want to have a transmitted wave for $x > a$ and a reflected one (+ the original wave) for $x < -a$? Is the the $E \to E + i\varepsilon$ prescription still correct? Obtain expression for the appriopriate Green's function and write integral equation for $\left\langle x \middle| \psi^{(?)} \right\rangle$.
   - Consider special case of an attractive ($\gamma > 0$) potential

   $$V = -\left(\frac{\gamma \hbar^2}{2m}\right) \delta(x).$$

   Solve the integral equation to obtain the transmission and reflection amplitudes.
   - The above potential has only one bound state (find it). Show that the transmission and reflection amplitudes have bound-state poles at the expected positions where k is regarded as complex.