Mechanika Kwantowa dla doktorantów
zestaw 19 na dzień 30.03.2017 godz. 8:15

1. Using expansion for the wave function calculated from the Schrödinger equation for a given potential $V(r)$
$$\left\langle \vec{r} \middle| \psi^{(+)} \right\rangle = \frac{1}{(2\pi)^{3/2}} \sum_l (2l+1) i^l A_l(kr) P_l(\cos\theta) \tag{1}$$
(which means that $A_l(r)$ are known) construct quantity β_l for some large enough R
$$\beta_l = \left. \frac{r}{A_l} \frac{dA_l}{dr} \right|_{r=R},$$
and by matching it with asymptotic form of $\left\langle \vec{r} \middle| \psi^{(+)} \right\rangle$ expressed in terms of phase shifts, show that
$$\tan\delta_l = \frac{kR\, j_l'(kR) - \beta_l\, j_l(kR)}{kR\, y_l'(kR) - \beta_l\, y_l(kR)}. \tag{2}$$
To this end use decomposition (prove it!):
$$A_l(kr) = \frac{1}{2}(e^{2i\delta_l}+1) j_l(kr) + i\frac{1}{2}(e^{2i\delta_l}-1) y_l(kr),$$
Derive general formula (in terms of spherical Bessel functions) for the phase shifts for the finite spherical well ($V_0 > 0$):
$$V(r) = \begin{cases} 0 & \text{dla} \quad R < r \\ -V_0 & \text{dla} \quad r < R \end{cases}.$$
In particular calculate $\tan\delta_0$. Discuss two limits $k \to 0$ and $k \to \infty$. How δ_0 depends on V_0?

2. Derive formula analogous to (2), but for f_l rather than for δ_l as a function β_l. To this end observe that
$$e^{2i\delta_l} = 1 + 2ik\, f_l$$
(here you will encounter Hankel function $h_l^{(+)} = h_l^{(1)} = j_l + iy_l$).

   Apply this formula to the square well from the previous problem and calculate f_0. Find possible poles of f_0.

3. Find energies of the bound states in spherical well from the first problem for $l = 0$. Depending on R and V_0 there is only a finite number of such states. Suppose that we tune V_0 in a continuous way. Then the energies of the bound states change (how?), and when V_0 increases new bound states appear for some discrete values of V_0^n ($n = 1, 2, 3\ldots$). Calculate V_0^n end the energy of the corresponding bound state. Show that V_0^n correspond to the singularities of f_0 for scattering energy $E \to 0$. Try to interpret this result.