Mechanika Kwantowa dla doktorantów
zestaw 17 na dzień 16.03.2017 godz. 8:15

1. Consider radial Schrödinger equation for a potential $V(r) \to 0$ for $r \to \infty$ and for energy $E = \hbar^2 k^2/(2m) > 0$ assuming that the wave function factorizes into a radial and angular parts
$$\psi_{klm}(\vec{r}) = R_{kl}(r) Y_l^m(\vartheta, \varphi)$$
where Y_l^m denotes spherical harmonics. For large r where $V(r) = 0$ (or equivalently for a free particle), this is so caled modified Bessel equation. First find solutions for $l = 0$ defining a new function $u(r) = r\, R_{k0}(r)$. Note that there are two possible solutions of this equation, one physical that is finite at the origin, and the other one that explodes for $r = 0$. They correspond to the spherical Bessel functions $j_l(kr)$ and $y_l(kr)$, respectively.

   Next, in order to find solutions for $l \neq 0$ do the following:

   - define $\chi_{kl}(r) = R_{kl}(r)/r^l$ and derive equation for $\chi_{kl}(r)$,
   - differentiate the above equation over r,
   - define a new function $f_{kl}(r) = r\, \chi'_{kl}(r)$,
   - compare equation for f_{kl} with the initial equation for χ_{kl} and read off the recurrence formula relating $\chi_{k\,l+1}$ to χ_{kl},
   - solve the requrrence formula for $R_{kl}(r)$ in the case $l = 1, 2$,
   - find asymptotic behaviour of $R_{kl}(r)$ for large r (that is find asymptotics for j_l but also for y_l),
   - using the initial equation for $R_{kl}(r)$ find asymptotic behaviour for small r (also both j_l and for y_l).

2. Show that for an infinite "hard ball"potential:
$$V(r) = \begin{cases} 0 & \text{dla} \quad a < r \\ \infty & \text{dla} \quad r < a \end{cases}$$
the radial wave function R_{kl}^{ball} differs from the one of a free equation R_{kl} by a phase shift $\delta_l(k)$
$$R_{kl}^{\text{ball}}(r) = R_{kl}(r + \delta_l(k)). \tag{1}$$
Calculate the phase shifts from the condition $R_{kl}(a) = 0$. Find low energy behaviour of $\delta_l(k)$.

   HINT: The solution for $l = 0$ is straightforward. However for $l \neq 0$ more work is needed. Use (1) in the asymptotic region $r \to \infty$ where one can use explicit asymptotic form of $j_l(kr+\delta_l(k))$. Use trigonometric identity to separate dependence on $\delta_l(k)$ and on rk. Replace asymptotic forms by full Bessel functions $j_l(kr)$ and $y_l(kr)$ and apply condition $R_{kl}(a) = 0$.

3. Prove by substituting to the spherical Bessel equation that

$$j_l(z) = \frac{1}{2}\frac{1}{i^l}\int_{-1}^{1} dt\, e^{izt} P_l(z).$$

To this end use the identity

$$P_l(z) = \frac{1}{2^l l!}\frac{d^l}{dz^l}(z^2 - 1)^l.$$