Mechanika Kwantowa dla doktorantów

zestaw 13 – 19.1.2017 at 8:15

1. Please read Chapter 7 in the book of Feynmam and Hibbs. Pay attention to the identity
$$\left\langle \frac{\delta F}{\delta x(s)} \right\rangle_S = -\frac{i}{\hbar} \left\langle F \frac{\delta S}{\delta x(s)} \right\rangle_S ,$$
that takes the following form in a discretized time
$$\left\langle \frac{\delta F}{\delta x_k} \right\rangle_S = -\frac{i}{\hbar} \left\langle F \frac{\delta S}{\delta x_k} \right\rangle_S ,$$
where
$$\langle F \rangle_S = \int [\mathcal{D}(x(t)] \; F[x(t)] \, e^{\frac{i}{\hbar} S[x(t)]}.$$
Using this identity show that for a particle in 3 dimensions one has:
$$< (x_{k+1} - x_k)^2 >=< (y_{k+1} - y_k)^2 >=< (z_{k+1} - z_k)^2 >= \frac{i\epsilon\hbar}{m},$$
$$< (x_{k+1}-x_k)(y_{k+1}-y_k) >=< (z_{k+1}-z_k)(y_{k+1}-y_k) >=< (x_{k+1}-x_k)(z_{k+1}-z_k) >= 0,$$
where we skipped subscript S in the brackets $\langle \ldots \rangle_S$.

2. Derive that
$$\lim_{\varepsilon \to 0} \langle \chi | \frac{m}{\varepsilon} (x_{k+1} - x_k) | \psi \rangle = \int dx \, \chi^*(x) \frac{\hbar}{i} \frac{\partial}{\partial x} \psi(x), \tag{1}$$
which defines momentum operator in a discretized time. Note that expression (1) spans over two adjacent time slices: t_{k+1} and t_k. So one can propagate wave functions ψ "forward" and χ "backwards"to t_k (or t_{k+1}). This means for example that
$$\langle \chi | x_{k+1} - x_k | \psi \rangle = \int dx \left[\chi^*(x, t+\varepsilon) \, x \, \psi(x, t+\varepsilon) - \chi^*(x,t) \, x \, \psi(x,t) \right].$$
Expanding wave functions in ε and using the Schrödinger equation one can derive (1).

3. Derive commutation relation for momentum and position operators calculating
$$\langle \chi | \, m \frac{1}{\varepsilon} \left((x_{k+1} - x_k) x_k - x_{k+1}(x_{k+1} - x_k) \right) | \psi \rangle ,$$
where
$$\langle \chi | F | \psi \rangle = \iint dx_1 dx_2 \, \chi^*(x_2) \left[\int_{x_1}^{x_2} [\mathcal{D}(x(t)] \; F[x(t)] \, e^{\frac{i}{\hbar} S[x(t)]} \right] \psi(x_1).$$
Follow the same steps as in problem 2.