Mechanika Kwantowa dla doktorantów

zestaw 12 – 12.1.2017 at 8:15

1. Finish problem 1 from the previus set by calculating integral (3).

   Instanton determinant. In this problem we will calculate explicitly the ratio of determinants
$$K' = \frac{\det'\left(-\frac{d^2}{d\tau^2} + V''(\bar{x}(\tau))\right)}{\det\left(-\frac{d^2}{d\tau^2} + 1\right)},$$
   where a double well potential $V(x)$ reads: $V(x) = \kappa(a^2 - x^2)^2$ with $\kappa = 1/(8a^2)$. Prime at the determinant means that the zero eigenvalue (zero mode) is not included, by $\bar{x}(\tau)$ we denote classical trajectory.

   - The eigenequation for quantum fluctuations around the classical trajectory (with $\tau_1 = 0$, where τ_1 is the time when the classical trajectory passes through zero):
$$\left[-\frac{d^2}{d\tau^2} + V''(\overline{x}(\tau))\right] y_n(\tau) = \lambda_n y_n(\tau) \tag{1}$$
     corresponds to the Schrödinger equation for a potential $U(\tau) = -3/(2\cosh^2(\tau/2))$ (where τ plays a role of a spacial variable) and energy $E_n = \lambda_n - 1$, which is discussed in the "Quantum Mechanics" of Landau and Lifischitz (probl. 5 page. 81 and probl. 4 page 88, Polish edition PWN 1979).

     Transform equation (1) into a hypergeometric equation for function w_n defined below:
$$y_n(\tau) = e^{\alpha\tau} w_n(\tau),$$
     where
$$\alpha = \pm\sqrt{-E_n}, \qquad E_n = \lambda_n - 1.$$
     Show that the solution reads:
$$y_n(\tau) = \mathcal{N}\left(3\tanh^2\left(\frac{\tau}{2}\right) - 6\alpha\tanh\left(\frac{\tau}{2}\right) + \left(4\alpha^2 - 1\right)\right)e^{\alpha\tau}.$$
     To this end introduce the following new variables:
$$\begin{aligned} z &= \tanh\left(\frac{\tau}{2}\right), \\ u &= \frac{1}{2}(1+z). \end{aligned}$$
     In terms of variable u Eq.(1) corresponds to the hypergeometric equation
$$u(1-u)\,w''(u) + \{c - (a+b+1)u\}\,w'(u) - ab\,w(u) = 0,$$
     whose solutions are given in terms of a series
$$w(u) = F(a,b,c;u) = 1 + \frac{ab}{c}\frac{u}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{u^2}{2!} + \dots$$

- Find discrete spectrum of the bound states ($E < 0$) for (1). Conditions that solutions vanish at $\tau = \pm\infty$ give quantization of α.
- To find contribution from the continuous spectrum we show first that there is no reflection for the particles scattering over the potential $U(\tau)$. To this end find asymptotics for two types of the solutions: $\alpha = ik$ and $\alpha = -ik$ in the limit $\tau \to \pm\infty$.
- If there is no reflection then the wave function $y_k(\tau)$ that asymptotically behaves as $e^{ik\tau}$ for $\tau \to \infty$, in the limit of $\tau \to -\infty$ behaves as $e^{ik\tau+i\delta_k}$, where δ_k is a phase shift. Show that

$$e^{i\delta_k} = \frac{1+ik}{1-ik}\frac{1+2ik}{1-2ik}.$$

  Identical argument applies to the wave function, that asymptotically behaves as $e^{-ik\tau}$.
- Close the system in a box $-T/2 < \tau < T/2$. Then the wave function inside the box is a superposition of two linearly independent solutions

$$y_n(\tau) = Ay_{\alpha=ik}(\tau) + By_{\alpha=-ik}(\tau)$$

  which vanishes at the boundaries

$$y_n(\pm T/2) = 0. \tag{2}$$

  If the box is large, it is enough to use asymptotic forms of $y_{\alpha=\pm ik}(\tau)$. Show that condition (2) leads to

$$Tk - \delta_k = \pi n.$$

  Let's denote solution to this equation by $\tilde{k}_n$. Similarly, for the Euclidean harmonic oscillator analogous solutions read $k_n = \pi n/T$.
- The contribution to K' coming from the continuous spectrum, K_{cont}, reads:

$$K_{cont} = \frac{\prod\tilde{\lambda}_n}{\prod\lambda_n} = \prod_{n=1}^{\infty}\frac{1+\tilde{k}_n^2}{1+k_n^2} = \exp\left(\sum_n \ln\frac{1+\tilde{k}_n^2}{1+k_n^2}\right) \approx \exp\left(\sum_n \frac{2k_n(\tilde{k}_n - k_n)}{1+k_n^2}\right).$$

- To calculate last sum under exponent go to the continuum limit $T \to \infty$ and convert the sum into the integral:

$$\ldots = \exp\left(\frac{1}{\pi}\int_0^{\infty} dk \frac{2\delta_k\, k}{1+k^2}\right) = \frac{1}{9}. \tag{3}$$

  Last equality can be obtained by integration by parts and the explicit form of δ_k. Full result for K' is obtained by multiplying K_{cont} by a non-zero λ value from the discrete part.

- Literature:

  S. Coleman, *Aspects of Symmetry*, Cambridge University Press (1988), Section 7, Appendix 1.

  A.I. Vainshtein, V.I. Zakharov, V.A. Novikov and M.A. Shifman, *ABC of Instantons*, Sov. Phys. Usp. **24**, 195 (1982) [Usp. Fiz. Nauk **136**, 553 (1982)].

  T. Schafer and E.V. Shuryak, *Instantons in QCD*, Rev. Mod. Phys. **70** (1998) 323 [arXiv:hep-ph/9610451].

2. Finding time dependence of the lowest eigenvalue $\lambda_1(T)$ of operator D_τ in the box $-T/2 < \tau < T/2$.

   Equation (1) for $\lambda_1 = 0$ has two linearly independent solutions $y_1(\tau)$ and $\tilde{y}_1(\tau)$ that have the following asymptotic behavior:

$$A e^{+i\omega\tau} \overset{\tau\to-\infty}{\longleftarrow} y_1(\tau) \overset{\tau\to+\infty}{\longrightarrow} A e^{-i\omega\tau}$$
$$-A e^{-i\omega\tau} \overset{\tau\to-\infty}{\longleftarrow} \tilde{y}_1(\tau) \overset{\tau\to+\infty}{\longrightarrow} A e^{+i\omega\tau}$$

   where $\omega = V''(\pm a)$. From these solutions we construct an antisymmetric quantity

$$B(\tau,\tau') = \tilde{y}_1(\tau)y_1(\tau') - \tilde{y}_1(\tau')y_1(\tau).$$

   Note that derivative $\partial_\tau B(\tau,\tau')|_{\tau'=\tau}$ is equal to the Wronskian. In the box the solution that satisfies boundary conditions

$$y_{\lambda=0}(-T/2) = 0, \qquad \partial_\tau y_{\lambda=0}(-T/2) = 1$$

   has a form

$$y_{\lambda=0}(\tau) = \frac{1}{2A\omega}\left(e^{T\omega/2}y_1(\tau) + e^{-T\omega/2}\tilde{y}_1(\tau)\right)$$

   Prove that differential equation

$$\left[-\frac{d^2}{d\tau^2} + V''(\overline{x}(\tau))\right] y_T(\tau) = \lambda_1(T) y_T(\tau)$$

   is equivalent to the integral equation

$$y_T(\tau) = y_{\lambda=0}(\tau) - \lambda_1(T)\int_{-T.2}^{\tau} d\tau' B(\tau,\tau')y_T(\tau'). \tag{4}$$

   In order to find time-dependenc of $\lambda_1(T)$ one has to demand that $y_T(T/2) = 0$. Assuming that for finite T eigenvalue $\lambda_1(T)$ is non-zero but small, show that

$$\lambda_1(T) = 4A^2 e^{-\omega T}.$$

3. Consider a particle of mass m moving in a potential $V(x)$. The potential takes the following form: it is an infinite potential well of length $2L$. Inside the well there is a symmetric barier of width $2a$ and height V_0. Since the potential is symmetric the wave functions are either symmetric or antisymmetric. Construct quantization conditions separately for symmetric and antisymmetric solutions. Assuming $m = \hbar = 1$, and taking some arbitrary numerical values for L, a and V_0 solve numerically (using e.g. Mathematica) quantization conditions for two lowest energy levels. Plot the wave functions and the probability density. Write time-dependent wave function that initially is concentrated in the left (or right) sub-well of the potential. Make an animated plot of time evolution of probability densities corresponding to such wave functions.