Mechanika Kwantowa dla doktorantów

zestaw 6 – 17.11.2016 at 8:15

1. Startując z 2 równań telegrafistów:

$$\frac{\partial P_\pm(x,t)}{\partial t} = a\left(P_\pm(x,t) - P_\mp(x,t)\right) \mp v\frac{\partial P_\pm(x,t)}{\partial x}$$

   wyprowadzić 2 równania 2 rzędu każde, jedno na P_+ drugie na P_-.

2. Show that for large N

$$\binom{N}{\mu}\left(\frac{1}{2}\right)^N \approx \sqrt{\frac{2}{\pi N}}e^{-\frac{j^2}{2N}}.$$

   Here $j = \mu - \nu$ and $N = \mu + \nu$ (see lecture notes on random walks).

3. Following the steps discussed at the lecture find $n = 2, 3, 4$ moments of a random walk probability distribution:

$$\mu_n(t) = \langle(\Delta x(t))^n\rangle$$

   where

$$\Delta x(t) = v\int_0^t d\tau\,(-)^{N(\tau)}$$

   with probability of $N(\tau)$ reversals equal r given by

$$\mathcal{P}(N(\tau) = r) = e^{-at}\frac{(at)^r}{r!}.$$

   The averaging over the number of reversals is understood in the following way:

$$\langle F(N(\tau)\rangle = \sum_{r=0}^{\infty} F(r)e^{-at}\frac{(at)^r}{r!}.$$

4. Show that the Laplace transform of the moments calculated in the previous problem is given by

$$\int_0^\infty dt\,e^{-st}\,\frac{\mu_m(t)}{m!\,v^m} = \begin{cases} \frac{1}{s^{(m+1)/2}}\frac{1}{(s+2a)^{(m+1)/2}} & \text{for odd } m \\ \frac{1}{s^{m/2+1}}\frac{1}{(s+2a)^{m/2}} & \text{for even } m\,. \end{cases}$$

5. Derive the Van Vleck formula

$$F(T) = \left(-\frac{1}{2\pi i\hbar}\frac{\partial^2 S}{\partial x\partial x_0}\right)^{1/2}$$

for the "quantum"part of the propagator

$$K = F(T) \exp \frac{i}{\hbar} S_{\text{cl}}$$

for one dimensional problem of a particle moving in potential V. Start from the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} K(x, x_0; t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] K(x, x_0; t),$$

write

$$K = \exp\left(\frac{i}{\hbar} S + \ln F + \dots \right)$$

and expand K in powers of $\hbar$. Show that in the first two orders in $\hbar$ the Schrödinger equation reduces to

$$\partial_t S + \frac{1}{2m} (\partial_x S)^2 + V(x) = 0, \tag{1}$$

and

$$\partial_t (\ln F) + \frac{1}{2m} \partial_x^2 S + \frac{1}{m} \partial_x S \partial_x (\ln F) = 0. \tag{2}$$

Differenciate (1) $\frac{\partial^2}{\partial x \partial x_0}$ and show that the equation obtained that way is identical to (2) where $\ln F$ has been replaced by $\frac{1}{2} \ln \frac{\partial^2 S}{\partial x \partial x_0}$ up to a constant that can be fixed from the normalization condition of the propagator for $t \to 0$:

$$K(x_b, x_a; t = 0) = \delta(x_b - x_a).$$