8.3 Transition amplitudes for charged particles

A charged particle moving in a given electromagnetic potential (in three dimensions) has, in general, the following Lagrangean:

$$L = \tfrac{1}{2}m\dot{\mathbf{r}}^2 - eV(\mathbf{r},t) + \frac{e}{c}\,\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)\,. \tag{8.60}$$

However, we are not interested in the potential V and set it equal zero. Some new interesting phenomena are generated by the vector potential $\mathbf{A}$. Therefore, we shall work with the following action:

$$S = S_0 + \sigma = \tfrac{1}{2}m\int dt\,\dot{\mathbf{r}}^2 + \frac{e}{c}\int dt\,\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)\,. \tag{8.61}$$

Following the notation explained earlier, the first order contribution to the perturbative expansion of the transition amplitude is

$$\frac{i}{\hbar}\,\langle\sigma\rangle_{S_0} = \frac{ie}{\hbar c}\left\langle\int dt\,\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)\right\rangle_{S_0}. \tag{8.62}$$

Now, we are dealing with an expression we have not encountered so far which needs to be precisely defined. Let us look at it after we discretize the time T which takes us from $\mathbf{x}$ to $\mathbf{y}$. We have

$$\sigma = \frac{e}{c}\int\limits_{C(\mathbf{x}\to\mathbf{y})} d\mathbf{r}\cdot\mathbf{A}(\mathbf{r},t) \tag{8.63}$$

where $C(\mathbf{x}\to\mathbf{y})$ stands for a Brownian path which takes us in time T from $\mathbf{x}$ to $\mathbf{y}$. But how do we write σ in a discretized form. Should we write

$$\sigma_a = \frac{e}{c}\sum_k(\mathbf{r}_{k+1}-\mathbf{r}_k)\cdot\mathbf{A}(\mathbf{r}_k,t_k) \tag{8.64}$$

or, perhaps, we should write

$$\sigma_b = \frac{e}{c}\sum_k(\mathbf{r}_{k+1}-\mathbf{r}_k)\cdot\mathbf{A}(\mathbf{r}_{k+1},t_{k+1})\ ? \tag{8.65}$$

Since, as we well know, $(\mathbf{r}_{k+1}-\mathbf{r}_k)/\epsilon$ is, in the language of the functional integrals, proportional to the momentum operator, the alternative (8.64) or (8.65) is equivalent to placing a differential operator before or after the function $\mathbf{A}$. Incidentally, one can convince oneself (do it!) that the difference between $\sigma_a - \sigma_b$ is of order one. This is so in the following sense: since this difference is going to end up as $<\sigma_b-\sigma_a>_{S_0}$, and one can employ the properties of the Brownian paths

$$\left\langle(x_{k+1}-x_k)^2\right\rangle_{S_0} = \frac{i\hbar\epsilon}{m} \quad \text{and} \quad \langle(x_{k+1}-x_k)(y_{k+1}-y_k)\rangle_{S_0} = 0\,,$$

we can write

$$\langle(\sigma_b-\sigma_a)\rangle \Rightarrow -\frac{e}{c}\sum_k\frac{\hbar\epsilon}{im}\left\langle\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{r}_k,t_k)\Big|_{\epsilon\to 0}\right\rangle = \left\langle-\frac{e}{c}\int dt\,\boldsymbol{\nabla}\cdot\mathbf{A}\right\rangle.$$

How do we settle this ambiguity? It was discussed by many, and there are standard references to the literature on the subject of mathematical questions of such limits as in (8.64) or (8.65), [8.1] and [8.2]. Our guide is to be in a correspondence with the Hamiltonian formulation. It turns out that the following prescription will do:

$$\langle\sigma\rangle_{S_0} = \frac{e}{c}\left\langle \sum_k \epsilon \frac{(\mathbf{r}_{k+1}-\mathbf{r}_k)}{\epsilon}\cdot\tfrac{1}{2}\left(\mathbf{A}(\mathbf{r}_{k+1},t_{k+1})+\mathbf{A}(\mathbf{r}_k,t_k)\right)\right\rangle_{S_0} . \quad (8.66)$$

This leads to an operator

$$V^{(1)} = -\frac{e}{2mc}\left(\mathbf{p}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{p}\right), \quad (8.67)$$

which is its first order equivalent in the Hamiltonian perturbation expansion.

In the next order we have

$$\begin{aligned}\frac{1}{2}\left(\frac{e}{\hbar c}\right)^2 &\left\langle\left(\int dt\,\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)\right)^2\right\rangle_{S_0} \\ = \frac{1}{2}\left(\frac{e}{\hbar c}\right)^2 &\left\langle \sum_k \epsilon\frac{(\mathbf{r}_{k+1}-\mathbf{r}_k)}{\epsilon}\cdot\tfrac{1}{2}(\mathbf{A}(\mathbf{r}_{k+1},t_{k+1})+\mathbf{A}(\mathbf{r}_k,t_k))\right. \\ &\left.\times\sum_l \epsilon\frac{(\mathbf{r}_{l+1}-\mathbf{r}_l)}{\epsilon}\cdot\tfrac{1}{2}(\mathbf{A}(\mathbf{r}_{l+1},t_{l+1})+\mathbf{A}(\mathbf{r}_l,t_l))\right\rangle_{S_0} . \quad (8.68)\end{aligned}$$

In the above expression the "off-diagonal" terms, $k \neq l$, give

$$\frac{e^2}{2mc}\left(\mathbf{p}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{p}\right)^2,$$

whereas the "diagonal" terms , $k = l$, provide

$$\frac{e^2}{2mc^2}\,\mathbf{A}\cdot\mathbf{A}\,.$$

Therefore, in the Hamiltonian formulation, the perturbation good to up to the second order is

$$V^{(1)+(2)} = -\frac{e}{2mc}\left(\mathbf{p}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{p}\right) + \frac{e^2}{2mc^2}\,\mathbf{A}\cdot\mathbf{A}\,. \quad (8.69)$$

It turns out that (8.69) is correct to all orders. Indeed, the exact Hamiltonian of a charged particle moving in the field **A** is

$$H = \frac{1}{2m}(\mathbf{p}-\frac{e}{c}\mathbf{A})\cdot(\mathbf{p}-\frac{e}{c}\mathbf{A}) = \frac{1}{2m}\,\mathbf{p}\cdot\mathbf{p} + V^{(1)+(2)}\,. \quad (8.70)$$

Let us give more arguments that (8.69) is the exact interaction potential: The Feynman argument which relates K to ψ can be repeated here. One can show that

$$\psi(\mathbf{y},T+\epsilon) = \psi(\mathbf{y},T) + \frac{e\epsilon}{2mc}\psi\boldsymbol{\nabla}\cdot\mathbf{A} - \frac{ie^2\epsilon}{2\hbar mc^2}\mathbf{A}^2\psi + \frac{i\epsilon\hbar}{2m}\boldsymbol{\nabla}^2\psi + \frac{e\epsilon}{mc}(\mathbf{A}\cdot\boldsymbol{\nabla})\psi$$

where

$$\psi(\mathbf{y}, T+\epsilon) = \int d^3x\, K(\mathbf{y}, \mathbf{x}, \epsilon)\, \psi(\mathbf{x}, T) \qquad \text{and} \qquad \frac{\epsilon}{T} \ll 1\,, \tag{8.71}$$

with $K(\mathbf{y}, \mathbf{x}, \epsilon)$ given by the action of (8.61):

$$K(\mathbf{y}, \mathbf{x}, \epsilon) = \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{\frac{3}{2}} e^{\frac{i}{\hbar}[m\frac{(\mathbf{y}-\mathbf{x})^2}{\epsilon} + \frac{e}{c}(\mathbf{y}-\mathbf{x})\cdot\mathbf{A}\left(\frac{1}{2}(\mathbf{x}+\mathbf{y}\right)]}\,. \tag{8.72}$$

One gets the result for $\psi(\mathbf{y}, T+\epsilon)$ through Taylor expansion and application of Gaussian integrals, keeping track of the leading order terms. (Note that the "midpoint dependence" of $\mathbf{A}$ on $\mathbf{x}$ and $\mathbf{y}$, which is equivalent to taking the average $\frac{1}{2}\left(\mathbf{A}(\mathbf{y}) + \mathbf{A}(\mathbf{x})\right)$, must be used to get the correct Schroedinger equation). Since

$$\psi(\mathbf{y}, T+\epsilon) - \psi(\mathbf{y}, T) \;=\; \epsilon\, \frac{\partial\psi}{\partial t}\,,$$

we get from the above expansion the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} \;=-\frac{\hbar^2}{2m}\nabla^2\psi \;+\; \frac{i\hbar e}{mc}(\mathbf{A}\cdot\boldsymbol{\nabla})\psi \;+\; \frac{i\hbar e}{2mc}\,\psi\boldsymbol{\nabla}\cdot\mathbf{A} \;+\; \frac{e^2}{2mc^2}\,\mathbf{A}^2\psi \tag{8.73}$$

which defines H as given in (8.70).

Now we shall comment on the complete amplitude for a charged particle moving in a given vector potential $\mathbf{A}$:

$$K(\mathbf{y}, \mathbf{x}, T) \;=\; \int\limits_{\mathbf{x}\to\mathbf{y}} [\mathcal{D}\mathbf{r}(t)]\, e^{\frac{i}{\hbar}\int_0^T dt\,(\frac{1}{2}m\dot{\mathbf{r}}^2 + \frac{e}{c}\dot{\mathbf{r}}\cdot\mathbf{A})}\,. \tag{8.74}$$

The first comment is on gauge invariance. The gauge transformation of the vector potential $\mathbf{A}$:

$$\mathbf{A}' \;=\; \mathbf{A} \;+\; \boldsymbol{\nabla} f\,, \tag{8.75}$$

where f is a scalar function, introduces into the exponent of (8.74) an additional term

$$\frac{ie}{\hbar c}\int\limits_0^T dt\, \dot{\mathbf{r}}\cdot\boldsymbol{\nabla} f \;=\; \frac{ie}{\hbar c}\int\limits_{\mathbf{x}}^{\mathbf{y}} d\mathbf{r}\cdot\boldsymbol{\nabla} f \;=\; \frac{ie}{\hbar c}\,[f(\mathbf{y}) - f(\mathbf{x})]\,. \tag{8.76}$$

Thus the new propagator becomes

$$K'(\mathbf{y}, \mathbf{x}, T) \;=\; \exp\left\{\frac{ie}{\hbar c} f(\mathbf{y})\right\}\, K(\mathbf{y}, \mathbf{x}, T)\, \exp\left\{-\frac{ie}{\hbar c} f(\mathbf{x})\right\}\,. \tag{8.77}$$

This expression determines the transformation of the wave functions under the gauge transformation (8.75), because we want to have the amplitude of transition from e.g. the initial state $\psi(\mathbf{x})$ to the final state $\chi(\mathbf{y})$ invariant under (8.75). Indeed we realise such an invariance provided the wave functions acquire a phase when (8.75) takes place:

$$\psi'(\mathbf{x}) \;=\; \exp\left\{\frac{ie}{\hbar c} f(\mathbf{x})\right\}\, \psi(\mathbf{x})\,. \tag{8.78}$$

Note that doing the integrations (8.76) along Brownian paths brings back the problem of the lack of uniqueness of this procedure (see the alternative (8.64) or (8.65)).

Let us look at this problem again from a somewhat different point of view, following [8.1]. For simplicity sake we shall do the calculations in *one dimension*. We start with the identity:

$$f(y) - f(x) \;=\; \sum_{j=0}^{N-1} \left[f(x_{j+1}) - f(x_j)\right] \tag{8.79}$$

with $x_N = y\,,\, x_0 = x$, and we *undo* this identity into a path integral of a derivative. For each j we define a new variable

$$\xi_\theta \;=\; x_j + \theta\,(x_{j+1} - x_j)\,, \qquad 0 < \theta < 1\,, \tag{8.80}$$

and expand

$$f(x_{j+1}) \;=\; f(\xi_\theta + (1-\theta)(x_{j+1} - x_j))$$

$$= f(\xi_\theta) \,+\, f'(\xi_\theta)(1-\theta)(x_{j+1} - x_j) \,+\, \tfrac{1}{2} f''(\xi_\theta)(1-\theta)^2 (x_{j+1} - x_j)^2 \,+\, \ldots\,,$$

and

$$f(x_j) \;=\; f(\xi_\theta - \theta\,(x_{j+1} - x_j))$$

$$= f(\xi_\theta) \,-\, f'(\xi_\theta)\,\theta\,(x_{j+1} - x_j) \,+\, \tfrac{1}{2} f''(\xi_\theta)\,\theta^2 (x_{j+1} - x_j)^2 + \ldots\;.$$

Therefore, (8.79) becomes

$$f(y) - f(x) \;=\; \sum_{j=0}^{N-1} \left[f(x_{j+1}) - f(x_j)\right] \tag{8.81}$$

$$= \sum_{j=0}^{N-1} \left[f'(\xi_\theta)(x_{j+1} - x_j) \,+\, \tfrac{1}{2} f''(\xi_\theta)(1-2\theta)(x_{j+1} - x_j)^2 \,+\, \mathcal{O}(x_{j+1} - x_j)^3\right].$$

Let us apply now (8.81) to integrations along quantal trajectories, hence along Brownian paths which are characterized by the diffusive relation between the steps in space Δx, executed during the time Δt:

$$(\Delta x)^2 = D \Delta t \tag{8.82}$$

where D is a constant of diffusion. Taking the continuous limit, i.e. $N \to \infty$ but keeping $|y - x|$ =const., we see that the first two terms in the expansion (8.81) are of the same (first) order while the third one is of the second order. So, only the first two terms survive the process of taking the limit, and we obtain an exotic formula due to Ito [8.1]

$$f(y) - f(x) \;=\; \int_x^y dx(t)\, \frac{df}{dx(t)} \,+\, \tfrac{1}{2}(1-2\theta) D \int_0^T dt\, \frac{d^2 f}{dx(t)^2}\,. \tag{8.83}$$

This formula reduces to the standard integral relation when we set $\theta = 1/2$, hence

$$\xi_{\frac{1}{2}} = \tfrac{1}{2}(x_{j+1} + x_j)\,. \tag{8.84}$$

This prescription of taking the "mid-point rule" (8.84) is equivalent to the previously employed rule of taking the average. More specifically: when there is an ambiguity of taking $A(x_{j+1})$ or $A(x_j)$ in a sum these two prescriptions are equivalent in the continuous limit

$$A(\tfrac{1}{2}\,(x_{j+1}+x_j)) \qquad \text{is equivalent to} \qquad \tfrac{1}{2}\,(A(x_{j+1})+A(x_j))\;. \tag{8.85}$$

An intuitive argument justifying (8.85) is to plot $A(x)$ versus x. When x_{j+1} and x_j approach each other, the two sides of (8.85) become the same.

The second comment (strictly speaking a group of comments) on the complete expression (8.74) is a discussion of its important special cases. The first example is the case of a constant magnetic field B directed along the z-axis. Then $\mathbf{A}$ lies in the (x, y)-plane

$$\mathbf{A} = \tfrac{1}{2}B\;[-y, x, 0]\;, \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \,=\, \hat{\mathbf{k}}\,B \tag{8.86}$$

where $\hat{\mathbf{k}}$ is the versor along the z-axis. Then the Lagrangian in (8.74) is a quadratic form

$$L \,=\, \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) \,+\, \frac{eB}{2c}(-\dot{x}y+\dot{y}x)\,, \tag{8.87}$$

and (8.74) can be evaluated exactly (as it was already done in one of exercises on Gaussian path integrals).

The second special case is of a magnetic flux line perpendicular to the (x, y)-plane. Now the vector potential we take in the following form

$$\mathbf{A}(x,y) \,=\, \frac{\kappa}{x^2+y^2}\;[-y, x, 0]\;. \tag{8.88}$$

Here $2\pi\kappa$ is the magnetic flux concentrated at the singularity $x = y = 0$,

$$2\pi\kappa \,=\, \oint d\mathbf{r}\cdot\mathbf{A} \,=\, \int d^2\mathbf{s}\cdot\boldsymbol{\nabla}\times\mathbf{A} \,=\, \int d^2\mathbf{s}\cdot\mathbf{B} \,=\, \Phi \tag{8.89}$$

where $d^2\mathbf{s}$ is the element of the plane perpendicular to $\mathbf{B}$. Indeed, calculating $\mathbf{B}$ from (8.88) we obtain

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \,=\, \hat{\mathbf{k}}\,\kappa\left(\partial_x\frac{x}{x^2+y^2} \,+\, \partial_y\frac{y}{x^2+y^2}\right) = 0\,. \tag{8.90}$$

Therefore $\mathbf{A}$ given by (8.88) represents *a flux tube with zero diameter but finite flux*. Incidentally, the integrals in (8.89) can be taken along the Brownian paths following e.g. the midpoint prescription, as we have indicated earlier.

The units of κ come naturally from the fact that the phase appearing in (8.74) must be dimensionless. This condition tells us that

$$\kappa \,=\, q\,\frac{\hbar c}{e} \tag{8.91}$$

where q is an arbitrary real number. Thus the magnetic flux is

$$\Phi = 2\pi\kappa = q\,\frac{hc}{e} = q \times 4.135 \cdot 10^{-7}\ \text{gauss} \cdot \text{cm}^2\,. \tag{8.92}$$

Our path integral (8.74) implies that, when our charged particle moves through space where a magnetic flux tube is present, *it is being influenced by this flux even if it is kept away from it and the magnetic field is zero along all allowed quantal trajectories.* This is so called Aharonov–Bohm effect [8.3]. When the diameter of the flux is not equal zero (which is the case in all real experiments), the restriction of possible quantal trajectories to these portions of space where $B = 0$ can be accomplished by a suitable arrangements of impenetrable screens.

Let us comment on the path integral with **A** given by (8.88) without any screening restrictions. Our initial point is **1** and the final one **2**. The motion is two dimensional, the origin of reference system coincides with the position of the singularity. We work out the phase of the path integral (8.74) acquired by going from **1** to **2** along a path C_{12} in the polar coordinates

$$x = r\cos\phi\,, \qquad y = r\sin\phi\,, \tag{8.93}$$

and

$$\int_{C_{12}} d\mathbf{r}\cdot\mathbf{A} = \kappa \int_{C_{12}} \frac{-ydx + xdy}{x^2+y^2}\,. \tag{8.94}$$

Since along the path

$$dx = \left(\frac{dr}{d\phi}\cos\phi - r\sin\phi\right)d\phi$$

$$dy = \left(\frac{dr}{d\phi}\sin\phi + r\cos\phi\right)d\phi\,,$$

and every time we go around the singularity we pick up the contribution $2\pi\kappa$ (compare (8.89)) we get for the phase we are looking for

$$\int\limits_{C_{12}} d\mathbf{r}\cdot\mathbf{A} = (\phi_2 - \phi_1 + 2\pi n)\,\kappa \tag{8.95}$$

where n is the number of times we encircled the singularity (one counterclockwise circulation brings $+1$, the circulation in the opposite sense gives -1. The number n is called the *winding number.* This winding number index appears when we have to deal with a plane punctured by the flux tube and has to be explicitly used in the process of summing over all trajectories, i.e. the sum over trajectories must involve $\sum_{n=0}^{\infty}$. For **A** given by (8.88) this summation can be analytically performed. We shall not do it here, one can find it in [8.4]

One can explicitly determine the magnetic phase in some specific experimental setups. The case in point is the Aharonov–Bohm effect we have already mentioned above. Indeed, let us imagine a *double slit experiment* modified in the following manner. In between the two slits, in the place which is protected from the incident beam by the screen, we place a very thin magnetic flux. It can be generated by a small solenoid whose magnetic field is completely confined to the interior of the solenoid and its flux precisely controlled. When it is placed very close to the screen, and the distance between the two slits is large enough, we can be virtually sure the electrons passing through slits have a negligible chance to hit the flux. Also with such a geometry there is no chance for electrons to circle the magnetic flux.

Let us take the initial point **1** at the source and the final point **2** at some place on the detection screen where we observe the diffractive pattern forming through interference of each electron passing through both slits. This interference comes from the difference in phase of the two amplitudes for passing through the two slits. When the magnetic flux is inserted in between the slits this difference in phase acquires an additional contribution equal

$$\Phi = q\,\frac{hc}{e}\,. \tag{8.96}$$

This can be seen as follows. Call the two slits (a) and (b), and take the two paths $C_{12}^{(a)}$, and $C_{12}^{(b)}$. From our earlier considerations we know that

$$\int\limits_{C_{12}^{(a)}} d\mathbf{r}\cdot\mathbf{A} + \int\limits_{C_{21}^{(b)}} d\mathbf{r}\cdot\mathbf{A} = \oint d\mathbf{r}\cdot\mathbf{A} = q\,\frac{hc}{e}\,. \tag{8.97}$$

But we also know that

$$\int\limits_{C_{21}^{(b)}} d\mathbf{r}\cdot\mathbf{A} = -\int\limits_{C_{12}^{(b)}} d\mathbf{r}\cdot\mathbf{A}\,, \tag{8.98}$$

therefore

$$\int\limits_{C_{12}^{(a)}} d\mathbf{r}\cdot\mathbf{A} = \int\limits_{C_{12}^{(b)}} d\mathbf{r}\cdot\mathbf{A} + q\,\frac{hc}{e}\,. \tag{8.99}$$

So, one can establish the Aharnov–Bohm effect (A-B) starting the observation of the diffractional pattern in the two-slit experiment *without magnetic flux in the solenoid*. Then we *increase the flux inside the solenoid* from zero and observe *the shift of the diffractive pattern produced on the screen*. This will happen (as we have already stressed) even though the charged particles never go through any magnetic field.

This truly amazing effect tells us that the electromagnetic *potentials* are the fundamental quantities in quantum mechanics, *not* the electromagnetic *fields*. It has been experimentally confirmed [8.5], [8.6]. We can also re-phrase it as follows. The A-B effect tells us that electrically charged particles interact through gauge fields **A** which are defined up to a gauge transformation (8.75). This, in turn, implies that the phases of transition amplitudes (and wave functions) can be arbitrarily chosen at any point of space-time. This is the simplest case of gauge invariance: the phases are arbitrary. On the other hand, traversing a closed loop (compare (8.97)) the phase is increased by Φ, and this *change of* *phase* is seen experimentally! This is the simplest example of gauge interactions which are (presumably) the backbone of all "fundamental interactions".

It is also very important to note that when q becomes an integer

$$q = n \qquad \text{hence} \qquad \kappa = n\,\frac{\hbar c}{e}\,, \quad n = 1, 2, 3, \ldots\,, \tag{8.100}$$

the A-B effect disappears in spite of the magnetic flux being different from zero. In other words in these cases charged particles "cease to see" the magnetic flux.

Let us go back to the situation where the only object in space is $\mathbf{A}$ given by (8.88). When $\kappa = n\hbar c/e$ the charged particles moving through space will not"see" the flux! (the trajectories of particles which hit this flux line, i.e. pass the origin, can be ignored because they are of measure zero). *So the propagator must be identical to the one of an empty space.*

One can look at it also this way. The phase factor in the propagator $K(\mathbf{y}, \mathbf{x}, T)$ is

$$\exp\left\{ i\frac{e}{\hbar c}\int_{C_{12}} d\mathbf{r}\cdot\mathbf{A} \right\} = \exp\left\{ i\frac{e}{\hbar c}(\phi_2 - \phi_1 + 2\pi n)\frac{\hbar c}{e} \right\} = \exp\left\{ i\frac{e}{\hbar c}(\phi_2 - \phi_1) \right\}\,,$$

which phase can be totally eliminated by a gauge transformation with $f(\mathbf{y}) = \phi_2$ and $f(\mathbf{x}) = \phi_1$. Thus we can insert into any stable (stationary) quantum mechanical system consisting of charged particles a flux

$$\Phi = 2\pi\kappa = n\,\frac{hc}{e}\,, \tag{8.101}$$

and the system *will not see it !*.

The well known example of a realization of this is the phenomenon of *quantization of magnetic fluxes inside superconductors.* The state of electrons in superconductors is, in a sense, given by just one wave function (usually called an order parameter, a complex function of the position). Now, in order to have a stable configuration of a magnetic flux penetrating the superconductor, this wave function (order parameter) must be single valued (i.e. must behave like a *bona fide* wave function). This is possible only when the magnetic flux is quantized in such a way as to become "invisible" in the sense of the A-B effect. Thus we arrive at the following quantization condition for a magnetic flux to exist inside a superconductor

$$\Phi = n\,\frac{hc}{e^*}\,, \qquad n = 1, 2, 3, \ldots \tag{8.102}$$

where e^* is an effective charge of quasiparticles which carry the current in the superconductors (so called "Cooper pairs" of electrons). This effect was predicted (long time ago!) by F. London [8.7], and then found in experiment [8.8].

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