Chapter 3

Random walks and their descendants

3.1 Differential (and Integral) Equations

In the preceding section we have shown that the random walk in which at each (discretized) time the particle has to move one step to the right or left with the same probability, 1/2, and whose probability distribution

$$P(j,N) = \binom{N}{\mu}\left(\frac{1}{2}\right)^N \approx \sqrt{\frac{2}{\pi N}}e^{-\frac{j^2}{2N}}\,, \qquad \text{with } j = 2\mu - N\,,$$

$$\underset{\text{continuum}}{\Longrightarrow} \sqrt{\frac{1}{4\pi Dt}}\; e^{-\frac{x^2}{4Dt}}\,, \tag{3.1}$$

leads in the continuum limit – on the one hand – to the diffusion equation (or the Euclidean version Schrödinger equation for a free particle), and – on the other hand – to the Smoluchowski integral equation. In this section we shall give examples of some other random walks and the equations they imply.

We start with the case which leads to the telegraph equations and its quantum mechanical *descendant*: Euclidean version of Dirac equation of free one dimensional motion [2.4], [3.1], [3.2]. Now the paths are composed of sections of the x-axis traversed by the particle at a constant velocity which is randomly reversed according to the following law:

- the probability of reversing the motion during the lapse of time dt is adt, where a is a constant,

- the probability of maintaining the same direction of motion during dt is cosequently $1 - a\,dt$.

Let $P_+(x,t)$ ($P_-(x,t)$) be the probability density of finding a particle movig to the right (left) at the position x and time t. For a small lapse of time Δt and the corresponding change of position Δx ($|\Delta x/\Delta t| = v =$ constant) we can write the following master equations for P_+ and P_-:

– the flow of particles to the right

$$P_+(x, t + \Delta t) = P_+(x - \Delta x, t)(1 - a\,\Delta t) + P_-(x + \Delta x, t)\,a\,\Delta t\,, \quad (3.2)$$

– the flow of particles to the left

$$P_-(x, t + \Delta t) = P_-(x + \Delta x, t)(1 - a\,\Delta t) + P_+(x - \Delta x, t)\,a\,\Delta t\,. \quad (3.3)$$

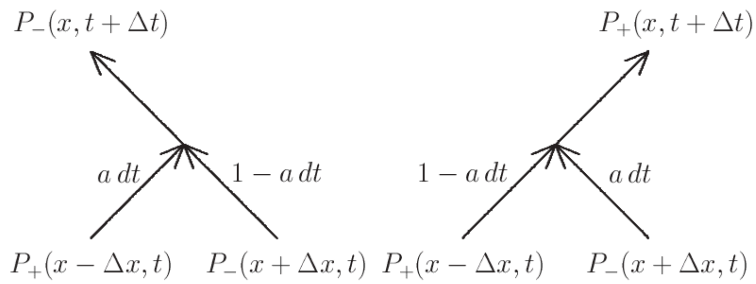

Figure 3.1: Probability densities of finding a particle movig to the left (right) at the position x and time $t + \Delta t$.

To get the differential equations for P_+ and P_- we expand equations (3.2,3.2) up to linear terms in Δx and Δt:

$$P_\pm(x \mp \Delta x, t)(1 - a\,\Delta t) \;\approx\; P_\pm(x,t) - P_\pm(x,t)\,a\,\Delta t \mp \Delta x\,\frac{\partial P_\pm(x,t)}{\partial x}\;,$$

$$P_\mp(x \pm \Delta x, t)\,a\,\Delta t \;\approx\; P_\mp(x,t)\,a\,\Delta t\;, \quad (3.4)$$

and obtain

$$\frac{\partial P_\pm(x,t)}{\partial t} \;=\; a\,(P_\mp(x,t) - P_\pm(x,t)) \mp v\,\frac{\partial P_\pm(x,t)}{\partial x}\,. \quad (3.5)$$

We can reduce these two coupled first order equations to two uncoupled second order equations, the same for each P

$$\frac{\partial^2 P_\pm}{\partial t^2} \;-\; v^2\,\frac{\partial^2 P_\pm}{\partial x^2} \;=\; -2a\,\frac{\partial P_\pm}{\partial t}\,. \quad (3.6)$$

This is the so called *telegraph equation* and gives a differential description of the random walk defined above. Note the relation between the telegraph equation and the diffusion equation: when $v \to \infty$, $a \to \infty$ but $2a/v^2$ is kept constant and equal to $1/D$, we obtain the diffusion equation.

As it turns out its first order version (3.5) is in one-to-one correspondence with the one dimensional Dirac equation. This equation we obtain following the routine steps of finding a first order equation

$$\left(i\hbar\frac{\partial}{\partial t} + i\hbar c\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - \beta mc^2\right)\psi = 0\,, \quad (3.7)$$

whose solutions satify the second order Klein–Gordon (or the so called "relativistic Schoedinger equation")

$$\left(-\hbar^2\frac{\partial^2}{\partial t^2}+\hbar^2c^2\nabla^2-m^2c^4\right)\psi=0\,. \tag{3.8}$$

To find $\boldsymbol{\alpha}$ and β we act on the r.h.s. of eq.(3.7) with the operator $i\hbar\partial/\partial t - i\hbar c\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\beta mc^2$, and find conditions which reduce the result to the Klein–Gordon equation. For $1+1$ dimensional spacetime with coordinates (t,x), this condition reads (for a complete four dimensional case see e.g. ref.[3.3]):

$$\left(-\hbar^2\frac{\partial^2}{\partial t^2}+\hbar^2c^2\alpha^2\frac{\partial^2}{\partial x^2}-m^2c^4\beta^2-mc^3(\alpha\beta+\beta\alpha)\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi=0\ . \tag{3.9}$$

Note that in this case we have only one matrix α. Clearly, α and β have to satisfy the following relations:

$$\alpha^2=\beta^2=1\,, \qquad\qquad \alpha\beta+\beta\alpha=0\,, \tag{3.10}$$

hence the choice, $\alpha=\sigma_z$ and $\beta=\sigma_x$, where σ_z and σ_x are the standard Pauli 2×2 matrices will do. So our two dimensional Dirac equation is

$$i\hbar\frac{\partial}{\partial t}\psi=-ic\hbar\,\sigma_z\frac{\partial}{\partial x}\psi+mc^2\sigma_x\psi \tag{3.11}$$

Note that ψ is a *two component* wave function. To compare (3.11) with (3.5) we write

$$\psi=e^{-i\frac{mc^2}{\hbar}t}\,u\,, \qquad\qquad u=\begin{pmatrix}u^+\\u^-\end{pmatrix}, \tag{3.12}$$

and insert it into (3.11). We get

$$\frac{\partial}{\partial t}u^\pm \;=\; \frac{imc^2}{\hbar}(u^\pm-u^\mp)\mp c\,\frac{\partial}{\partial x}u^\pm\,. \tag{3.13}$$

Let us now continue (3.5) to the Euclidean time: $t\to -i\tau$, $v\to iv$, (or, equivalently, $m\to im$ in eq.(3.13)), and rename $a=+mc^2/\hbar$ and $v=c$ (both are velocities). We obtain:

$$\frac{\partial P_\pm}{\partial\tau}=\frac{mc^2}{\hbar}(P_\pm-P_\mp)\mp c\frac{\partial P_\pm}{\partial x}\,. \tag{3.14}$$

So, our telegraph equations are, after continuation to the Euclidean time, isomorfic with the two component, one dimensional, Dirac equation.

One can make the following comments. The wave function of the Dirac equation (3.11) can be obtained as a sum over trajectories constructed from random walks leading to the telegraph equation. To construct the paths we may, for instance, write the final $P_+(x,t)$ in terms of the initial $P_+(x-k\Delta x, t-n\Delta t)=1$ (where k is the number of steps to the right) going backwards n steps in time, with the help of (3.2, 3.3). When $n\gg 1$ there are many paths to do it, and we can write $P_+(x,t)$ as a sum over all these paths:

$$P_+(x,t)=\sum_{\text{paths}}(1-a\Delta t)^{n-r}(a\Delta t)^r \tag{3.15}$$

where r is the number of reversals of the directions along the path. When the time steps are very small, we can set $1 - a\Delta t \approx 1$. Going back to the real time, $a\Delta t = i\epsilon$, we write the Feynman propagator for a free relativistic particle in one space and one time dimension

$$K(b,a) \approx \sum_r N(r)(i\epsilon)^r \tag{3.16}$$

where $N(r)$ is the number of paths with r reversals. This is the formula given in ref.[1.7], Problem 2-6.

All this was written in a discretized form. What about the continuous limit, which was so easy for the Brownian random walk? In the case of the Poisson process we are dealing with now, finding the probability distribution for positions x, $P(x(t))$, is more complicated. By definition a Poisson process in time, t, is the one in which the probability of its occurence (in our case: of the "reverse") in $(t, t + dt)$ is adt , and the probability of no occurence (no "reverse") is $1 - adt$. Thus (see e.g. ref.[2.5]) the probability of $N(t)$ "reverses" in time $(0, t)$ is given by the Poisson formula

$$\text{Prob}(N(t) = r) \; = \; e^{-at}\,\frac{(at)^r}{r!}\,, \qquad\qquad r = 0, 1, 2, 3, ... \tag{3.17}$$

and, for the sequence of ordered times $t_1 < t_2 < t_3 < ... < t_n$, the increments $N(t_2) - N(t_1), N(t_3) - N(t_2), \ldots, N(t_n) - N(t_{n-1})$ are *statistically independent.* Thus e.g. $N(t_2) - N(t_1)$ is the stochastic variable of the number of reverses in time $t_2 - t_1$ governed by the Poisson formula.

In order to find a probability distribution in x ,$P(x)$, we first define $G(k)$, the generating function for the moments

$$G(k) = \int_{-\infty}^{+\infty} dx\, e^{ikx} P(x) = \sum_{m=0}^{\infty} \frac{(ik)^m}{m!}\mu_m \tag{3.18}$$

where

$$\mu_m = \int_{-\infty}^{+\infty} dx\, x^m P(x) \tag{3.19}$$

are the moments of the position random variable. Having the moments we have $G(k)$, and inverting (3.18) we get the distribution $P(x)$:

$$P(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, G(k)\, e^{-ikx}\,. \tag{3.20}$$

So, let us work out the moments of the random variable $x(t)$. The velocity as a function of time is expressed through the random variable $N(t)$

$$v(t) = v\,(-1)^{N(t)} \tag{3.21}$$

where v is the constant which appeared in eq.(3.5). Let $\Delta x = x_2 - x_1$ where x_1 and x_2 are the initial and final positions of the particle. The distance travelled in time t is

$$\Delta x(t) = \int_0^t d\tau\, v(\tau) = v \int_0^t d\tau (-1)^{N(\tau)} . \tag{3.22}$$

Note that $P_+ = P(\Delta x(t))$, because $\text{Prob}(N(t) = r)$ of (3.17) starts with being 1 for $at \ll 1$ and $r = 0$, hence $\Delta x(t)$ increases and the particle is moving to the right. Similarly, $P_- = P(-\Delta x(t))$ because now the particle is moving to the left.

Since we know the probability distribution of $N(t)$, we can compute the average distance travelled in time t (hence the first moment of the random variable $\Delta x(t)$)

$$\begin{aligned} \langle \Delta x(t) \rangle = \mu_1(t) = v \left\langle \int_0^t d\tau (-1)^{N(\tau)} \right\rangle = v \int_0^t d\tau \left\langle (-1)^{N(\tau)} \right\rangle \\ = v \int_0^t \sum_{r=0}^{\infty} (-1)^r e^{-a\tau} \frac{(a\tau)^r}{r!} d\tau = v \int_0^t d\tau\, e^{-a\tau} e^{-a\tau} = v \int_0^t d\tau\, e^{-2a\tau} . \end{aligned} \tag{3.23}$$

In order to see how the higher moments can be evaluated let us calculate the second (and higher) moments.

$$\begin{aligned} \mu_2(t) \;=\; \left\langle \Delta x^2(t) \right\rangle \;=\; v^2 \left\langle \left(\int_0^t d\tau\, (-1)^{N(\tau))} \right)^2 \right\rangle \\ = v^2 \left\langle \int_0^t d\tau_1 \int_0^t d\tau_2\, (-1)^{N(\tau_1)} (-1)^{N(\tau_2)} \right\rangle . \end{aligned} \tag{3.24}$$

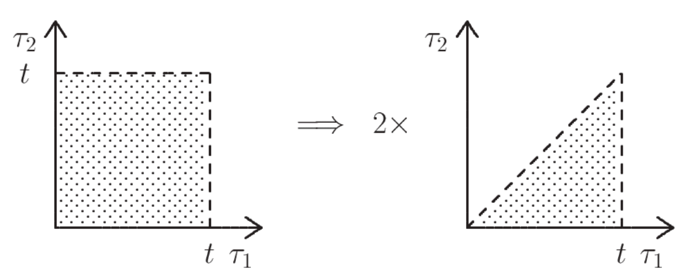

Figure 3.2: Change of integration described in the text.

The integration is over the square ($0 < \tau_1 < t$, $0 < \tau_2 < t$). Since the integrand is symmetric in τ_1 and τ_2, it is convenient to replace it by the interaction over the triangle: $0 \le \tau_1 \le \tau_2 \le t$. Thus we have

$$\mu_2(t) \,=\, \langle \Delta x^2(t) \rangle \,=\, 2!\, v^2 \left\langle \int\int_{0\le\tau_1\le\tau_2\le t} d\tau_1 d\tau_2\, (-1)^{N(\tau_1)+N(\tau_2)} \right\rangle$$
$$= 2!\, v^2 \left\langle \int\int_{0\le\tau_1\le\tau_2\le t} d\tau_1 d\tau_2\, (-1)^{N(\tau_2)-N(\tau_1)} \right\rangle \tag{3.25}$$

where we employed the identities: $N(\tau_2) = N(\tau_1) + [N(\tau_2) - N(\tau_1]$, and $(-1)^{2N(\tau_1)} = 1$. Since $N(\tau_2) - N(\tau_1)$ is the random variable for the number of reverses in the lapse of time $\tau_2 - \tau_1$ we can again take the average as in (3.23) and have

$$\left\langle (-1)^{N(\tau_2)-N(\tau_1)} \right\rangle \,=\, \sum_{r=0}^{\infty} (-1)^r e^{-a(\tau_2-\tau_1)}\, \frac{a(\tau_2-\tau_1)^r}{r!} = e^{-2a(\tau_2-\tau_1)}\,. \tag{3.26}$$

Thus μ_2 can be presented as a convolution

$$\mu_2(t) \,=\, 2!\, v^2 \int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1\, e^{-2a(\tau_2-\tau_1)}$$
$$= 2!\, v^2 \int_0^t d\tau_2\; \theta(t-\tau_2) \int_0^{\tau_2} d\tau_1\, \theta(\tau_2-\tau_1)\, e^{-2a\tau_1} \tag{3.27}$$

where θ functions are not redundant, because they make possible an explicit presentation of $\mu_2(t)$ as a double convolution.

Note that to write the first moment as a single convolution we also need θ function:

$$\mu_1(t) = v \int_0^t d\tau\, e^{-2a\tau} = v \int_0^{\infty} d\tau\, \theta(t-\tau)\, e^{-2a\tau}\,. \tag{3.28}$$

In fact, with the help of θ functions we can present any moment $\mu_m(t)$ as a m-fold convolution. The exercise below of working out $\mu_3(t)$ and $\mu_4(t)$ will tell us all we need to know about the moments (there is a difference between m-odd and m-even moments).

Exercise

First $\mu_3(t)$. We follow similar steps as in the case of $\mu_2(t)$

$$\mu_3(t) = v^3 \left\langle \int_0^t \int_0^t \int_0^t (-1)^{N(\tau_1)}(-1)^{N(\tau_2)}(-1)^{N(\tau_3)}\, d\tau_1 d\tau_2 d\tau_3 \right\rangle$$

$$= 3!\, v^3 \int_{0<\tau_1<\tau_2<\tau_3<t} d\tau_1 d\tau_2 d\tau_3 \left\langle (-1)^{N(\tau_1)+N(\tau_2)+N(\tau_3)} \right\rangle .$$

To introduce independent random variables we use the identity

$$N(\tau_1) + N(\tau_2) + N(\tau_3) = 2N(\tau_2) + [N(\tau_3) - N(\tau_2)] + [N(\tau_1) - N(0)] .$$

(Remember that $N(0) = 0$). Thus

$$\frac{\mu_3(t)}{3!\, v^3} = \int_{0<\tau_1<\tau_2<\tau_3<t} d\tau_1 d\tau_2 d\tau_3 \left\langle (-1)^{[N(\tau_3)-N(\tau_2)]} \right\rangle \left\langle (-1)^{[N(\tau_1)-N(0)]} \right\rangle$$

$$= \int_0^\infty \theta(t-\tau_3)\, d\tau_3\, \theta(\tau_3-\tau_2)\, e^{-2a(\tau_3-\tau_2)} d\tau_2\, \theta(\tau_2-\tau_1)\, d\tau_1\, e^{-2a\tau_1}$$

where we inserted $d\tau$' s to bring out the structure of the triple convolution.

Now $\mu_4(t)$,

$$\mu_4(t) = v^4 \left\langle \int_0^t \int_0^t \int_0^t (-1)^{N(\tau_1)}(-1)^{N(\tau_2)}(-1)^{N(\tau_3)}(-1)^{N(\tau_4)} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \right\rangle$$

$$= 4!\, v^4 \int_{0<\tau_1<\tau_2<\tau_3<\tau_4<t} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \left\langle (-1)^{N(\tau_1)+N(\tau_2)+N(\tau_3)+N(\tau_4)} \right\rangle .$$

Now we introduce the independent random variables

$$N(\tau_1) + N(\tau_2) + N(\tau_3) + N(\tau_4) =$$

$$= 2N(\tau_3) + [N(\tau_4) - N(\tau_3)] + 2N(\tau_1) + [N(\tau_2) - N(\tau_1)] .$$

Thus

$$\frac{\mu_4(t)}{4!\, v^4} = \int_{0<\tau_1<\ldots<\tau_4<t} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \left\langle (-1)^{[N(\tau_4)-N(\tau_3)]} \right\rangle \left\langle (-1)^{[N(\tau_2)-N(\tau_1)]} \right\rangle$$

$$= \int_0^\infty \theta(t-\tau_4)\, d\tau_4\, \theta(\tau_4-\tau_3)\, e^{-2a(\tau_4-\tau_3)}$$

$$\times\, d\tau_3\, \theta(\tau_3-\tau_2)\, d\tau_2\, \theta(\tau_2-\tau_1)\, e^{-2a(\tau_2-\tau_1)}\, d\tau_1\, \theta(\tau_1)$$

where again, we inserted $d\tau$'s to bring out the structure of the quadruple convolution.

End of Exercise

The reason for representing $\mu_m(t)$ as m-fold convolutions is to find an analytic form of the probability density $P(\Delta x(t))$ through its Laplace transform

$$\Pi(x_2 - x_1, s) = \int_0^\infty dt\, e^{-st} P(x_2 - x_1, t) \tag{3.29}$$

where $P(x_2 - x_1, t) = P(\Delta x(t))$, with $\Delta x(t)$ given by (3.22).

Indeed, as is well known, the Laplace transform of an m-fold convolution is simply a product of the Laplace transforms of its m components. Since only two functions appear in the convolutions, we need only two Laplace transforms

$$\int_0^\infty d\tau\, \theta(\tau)\, e^{-\tau s} = \frac{1}{s} \qquad \text{and} \qquad \int_0^\infty d\tau\, \theta(\tau)\, e^{-\tau(s+2a)} = \frac{1}{s+2a}\,, \tag{3.30}$$

and from the special cases worked out above we can deduce

$$\int_0^\infty dt\, e^{-st}\, \frac{\mu_m(t)}{m!\, v^m} = \begin{cases} \frac{1}{s^{(m+1)/2}} \frac{1}{(s+2a)^{(m+1)/2}} & \text{for odd } m \\ \frac{1}{s^{m/2+1}} \frac{1}{(s+2a)^{m/2}} & \text{for even } m\,. \end{cases} \tag{3.31}$$

So, from (3.20), (3.29) and (3.31) we can write down the complete expression for the Laplace transform of $P(x_2 - x_1, t)$:

$$\Pi(x_2 - x_1, s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{-ik(x_2-x_1)} \sum_{m=0}^{\infty} (ikv)^m \times \begin{cases} \frac{1}{s^{(m+1)/2}} \frac{1}{(s+2a)^{(m+1)/2}} & \text{for odd } m \\ \frac{1}{s^{m/2+1}} \frac{1}{(s+2a)^{m/2}} & \text{for even } m\,. \end{cases} \tag{3.32}$$

The sum in (3.32) can be evaluated:

$$\begin{aligned} \sum_{m\,odd} (ikv)^m \frac{1}{[s(s+2a)]^{(m+1)/2}} &+ \sum_{m\,even} (ikv)^m \frac{1}{s} \frac{1}{[s(s+2a)]^{m/2}} \\ &= \sum_{l=0}^{\infty} \left(\frac{ikv}{s(s+2a)} \frac{(ikv)^{2l}}{[s(s+2a)]^l} + \frac{1}{s} \frac{(ikv)^{2l}}{[s(s+2a)]^l} \right) \\ &= \left(\frac{1}{s} + \frac{ikv}{s(s+2a)} \right) \frac{1}{1 - \frac{(ikv)^2}{s(s+2a)}} \end{aligned}$$

where we performed (formally) the sum of a geometric series. Thus we get an analytic expression for Π:

$$\Pi(x_2 - x_1, s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{-k(x_2-x_1)} \left(\frac{1}{s} + \frac{ikv}{s(s+2a)} \right) \frac{1}{1 - \frac{(ikv)^2}{s(s+2a)}}\,. \tag{3.33}$$

We can also write down $P(x_2 - x_1, t)$ as an inverse Laplace transform of (3.33): when there exists a real constant a_0 such that the integral

$$\int_0^\infty dt\, e^{-a_0 t}\, |P(x_2 - x_1, t)|$$

exists, then

$$\begin{aligned} P(x_2 - x_1, t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{-k(x_2 - x_1)} \\ &\times \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} ds\, e^{st} \left(\frac{1}{s} + \frac{ikv}{s(s+2a)} \right) \frac{1}{1 - \frac{(ikv)^2}{s(s+2a)}} \end{aligned} \tag{3.34}$$

where $a \geq a_0$, otherwise arbitrary.

Several comments are now in order. We have discussed a random walk in one spatial dimension assuming it to be a Poissonian process. It turned out that the corresponding probability densities satisfy the telegraph equations (3.5) which turn out to be an Euclidean form of the Dirac equation for a free particle (of a finite mass) moving in one spatial dimension (3.13). We also performed the sum over all paths and obtained a (formal) expression for the probability densities (hence the Euclidean propagators for the Dirac equation). The essential ingredient in this equivalence is a different from zero mass of the Dirac particle: *for a massless Dirac particle there is no Poisson process.*

An obvious (and important) question one may ask is whether all these procedures can be generalized to 3+1 dimensions. This problem was addressed in ref.[3.1]. It turns out that

(a) one can define a Poissonian process in 3+1 dimensions and have telegraph-like equations for the Dirac amplitudes u_+, u_-

$$\frac{\partial u_\pm}{\partial t} = \frac{imc^2}{\hbar}(u_\pm - u_\mp) \mp c\,\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} u_\pm \tag{3.35}$$

where, as before, $a = mc^2/\hbar$. Note that the one dimensional operator $c\partial/\partial x$ is replaced by the three dimensional one $c\,\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}$, where $\boldsymbol{\sigma}$ are the three Pauli matrices. Therefore both u_+ and u_- become two component spinors.

(b) The trouble is however in presenting the spinors $u_\pm$ as sums over paths. The procedure described above for the case of 1+1 dimensions can be, in principle, generalized into a "spinor chain path integral" for the Dirac equation, but its relation to the underlying stochastic process (related by the analytic continuation) is not clear.

(c) In fact, it is very likely that this intuitive approach to the propagation of the Dirac particles described here is – strictly speaking – untenable and one has to recourse to such horrors as the Grassman algebra and an adaptation of the spinor calculus to the Euclidean metric.

(d) There is an interesting appendage to the above (see ref.[3.2]). The authors introduce four, instead of two, probability distributions: the probabilities of moving left or right in space while moving forwards or *backwards* in time. One can, similarly as above, construct a set of master equations which result in the Dirac equation without recourse to the continuation from the Euclidean to the real time. So [3.2] shows that the random motion of a particle in one space- and one time dimension leads directly to the Dirac equation *provided backward in time motions are introduced.*

References

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