Chapter 1

Path integrals and superfluidity

What happens when we go below T_c? The function $\mu(T)$ cannot go positive, hence we keep it zero. We can keep $\langle N \rangle$ fixed (as $T \to 0$) only *through condensation of particles at the lowest (zero energy) level.* For $T < T_c$, the average number of particles occupying levels other than the ground level,

$$\langle N_{\neq 0} \rangle = V \left(\frac{m k_B T}{2\pi\hbar^2} \right)^{\frac{3}{2}} 2.612 = V \int \frac{d^3 p}{(2\pi\hbar)^3} \left(e^{\frac{p^2}{2 m k_B T}} - 1 \right)^{-1} , \qquad (1.33)$$

decreases as $T^{3/2}$ feeding the ground level to keep $\langle N \rangle$ fixed. From (**??**), (**??**) and (1.33) we get for the number of particles at the ground level the following expression

$$\langle N \rangle_0 = \langle N \rangle - \langle N_{\neq 0} \rangle = \langle N \rangle \left(1 - \left(\frac{T}{T_c} \right)^{\frac{3}{2}} \right) . \qquad (1.34)$$

So, at $T = 0$ all particles condense at the lowest level.

1.1 Condensations and critical points

In order to appreciate this "criticality" of behavior of a system of free Bosons described above let us compare it with a system of mutually noninteracting Bosons which can reside only on some well defined discrete sequence of states whose energies are $\epsilon_i > 0$, $\epsilon_{i+1} > \epsilon_i$, and where $0 \leq i \leq M$, M being a finite natural number.

The average number of Bosons, N, which we will keep, as before, fixed is the discrete version of the continuous distribution (**??**).

$$N = \sum_{i=0}^{M} \frac{1}{e^{\beta(\epsilon_i - \mu(T))} - 1} = \frac{1}{e^{\beta(\epsilon_0 - \mu(T))} - 1} + \sum_{i=1}^{M} \frac{1}{e^{\beta(\epsilon_i - \mu(T))} - 1} \qquad (1.35)$$

where, as before, $\beta = 1/k_B T$. We can write (1.35) as a sum of the occupation numbers of the states i:

$$N = N_0 + \sum_{i=1}^{M} N_i \, . \qquad (1.36)$$

Here we set apart the contribution to the fixed average number of particles, N, of the ground level (ϵ_0) and and of the remaining levels ($\epsilon_i,\ i \neq 0$).

Now we let $T \to 0$ varying $\mu(T)$ in such a way as to keep N fixed - the same process we have done for the system of free Bosons. Clearly the specific form of the function $\mu(T)$ depends on the assumed values for ϵ_i, but - in any case - we have to keep $\mu(T)$ increasing towards ϵ_0. However, unlike in the previous case, *we shall not encounter any special temperature in this process*. As the temperature drops, Bosons will flow from higher levels to the lowest level and, eventually, $\sum_{i=1}^{M}$ will become negligible and only the first term (corresponding to ϵ_0) will carry all of them.

Note that, without loosing generality of arguments, we can shift the positions of the levels: $\epsilon_i \to \epsilon_i - \epsilon_0, \quad \epsilon_0 \to 0$. Then

$$N \;=\; \frac{1}{e^{-\beta\mu(T)} - 1} \;+\; \sum_{i=1}^{M} \frac{1}{e^{\beta(\epsilon_i - \mu(T))} - 1}\,, \tag{1.37}$$

and now we must have $\mu(T) \leq 0$.

By letting $\mu(T)$ approach zero (from below) we can keep N fixed. Since $\epsilon_i > 0$ and $\mu(T) \leq 0$, the sum $\sum_{i=1}^{M}$ equals zero at $T = 0$. N is then taken care of by N_0, and we have to assume that for T close to zero, $\mu(T) = \lambda k_B T$. Then, from (1.37) we get

$$\lim_{T\to 0}\left(-\frac{\mu(T)}{k_B T}\right) = \lambda\,, \qquad\qquad \lambda = \ln\frac{N+1}{N}\,. \tag{1.38}$$

There is nothing surprising there: when we cool this system all particles end up at the lowest level.

We have to contrast this behavior with the one of free Bosons: there the lowest level stays empty until we reach T_c, then it starts taking Bosons and completes this process at $T = 0$. This critical temperature depends, for a fixed density of Bosons, only on their mass and fundamental constants:

$$T_c \;=\; \frac{2\pi\hbar^2}{m k_B}\left[\frac{1}{2.612}\frac{\langle N\rangle}{V}\right]^{2/3}. \tag{1.39}$$

Can we have $T_c \neq 0$ for a discrete spectrum? The answer is no. Indeed, let us suppose that we have $T_c \neq 0$. Then $\mu(T_c)$ obtained from the equation:

$$N \;=\; \sum_{i=1}^{M} \frac{1}{e^{\beta(\epsilon_i - \mu(T_c))} - 1}$$

would have to give

$$N_0 \;=\; \frac{1}{e^{-\frac{\mu(T_c)}{k_B T_c}} - 1} \;=\; 0\,,$$

which , in turn, would imply

$$e^{-\frac{\mu(T_c)}{k_B T_c}} \;=\; +\infty \qquad \text{hence} \qquad -\mu(T_c) \;=\; +\infty\,,$$

because $T_c \neq 0$. But this would make $N = 0!$ So, indeed, for a discrete spectrum we cannot have a well defined onset of condensation.

We can look at this problem of how the spectrum influences T_c working out the case of Bosons in a harmonic trap: all Bosons are confined by an external harmonic potential

$$V(\mathbf{r}) = \tfrac{1}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \; . \tag{1.40}$$

We discuss anisotropic traps because they can be realized in experiments and analysed theoretically [11.5]. In fact some Bosonic condensates have recently been observed for groups of atoms ranging from a few thousands to several milions [11.6]. Note that by doing this we give a finite size to the system which is

$$a_0 = \sqrt{\frac{\hbar}{m\omega_0}} \, , \qquad \omega_0 = (\omega_x \omega_y \omega_z)^{1/3} \, . \tag{1.41}$$

Since each Boson moves in the potential (1.40) its energy levels are

$$\epsilon_{n_x n_y n_z} \; = \; (n_x + \tfrac{1}{2})\hbar\omega_x + (n_y + \tfrac{1}{2})\hbar\omega_y + (n_z + \tfrac{1}{2})\hbar\omega_z \, . \tag{1.42}$$

All statements we have formulated earlier on discrete spectra, ϵ_i, apply here. We have

$$N \; = \; \frac{1}{e^{\beta(\epsilon_{000} - \mu(T))} - 1} \; + \sum_{n_x, n_y, n_z \neq 0} \frac{1}{e^{\beta(\epsilon_{n_x n_y n_z} - \mu(T))} - 1} \, , \tag{1.43}$$

and, as long as the average distance between levels, $\hbar(\omega_x \omega_y \omega_z)^{1/3}$, is substantially larger than $k_B T$ we have a well defined *discrete* spectrum, and the critical temperature for condensation is not well defined.

However, the spectrum (1.42) becomes, for all practical purposes, *continuous* when $k_B T$ is much larger than the average distance between levels

$$k_B T \; \gg \; \hbar(\omega_x \omega_y \omega_z)^{1/3} = \hbar\omega_0 \, . \tag{1.44}$$

As it turns out [11.6], the critical temperatures, T_c, of the observed condensates do satisfy (1.44). In the available traps one observes $k_B T \simeq (20 - 200)\hbar\omega_0$ with $\hbar\omega_0$ being of the order of a few nK. Therefore, it makes good sense to take the continuous limit of the oscillator spectrum and find T_c defined by such a system.

It follows from our earlier discussion that

$$\mu(T_c) \; = \; \epsilon_{000} \; = \; \tfrac{1}{2}\hbar(\omega_x + \omega_y + \omega_z) \, , \tag{1.45}$$

and the ground state should be separated out from the rest of the spectrum because its contribution to N,

$$N \; = \; N_0 \; + \int_0^\infty \frac{dn_x dn_y dn_z}{e^{\beta\hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z)} - 1} \, , \tag{1.46}$$

in the continuous sum is zero. We follow the same steps as in our earlier discussion of free Bosons (1.33): we get T_c assuming that $N_0(T_c) = 0$. Thus we determine T_c from the equation

$$
\begin{aligned}
N &= \int_0^\infty \frac{dn_x dn_y dn_z}{e^{\beta\hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z)} - 1} = \left(\frac{k_B T_c}{\hbar\omega_0}\right)^3 \int_0^\infty \frac{dxdydz}{e^{x+y+z} - 1} \\
&= \left(\frac{k_B T_c}{\hbar\omega_0}\right)^3 \int_0^\infty dxdydz \, \frac{e^{-x}e^{-y}e^{-z}}{1 - e^{-x}e^{-y}e^{-z}} \\
&= \left(\frac{k_B T_c}{\hbar\omega_0}\right)^3 \int_0^\infty dxdydz \left(e^{-x}e^{-y}e^{-z} + e^{-2x}e^{-2y}e^{-2z} + ...\right) \\
&= \left(\frac{k_B T_c}{\hbar\omega_0}\right)^3 \zeta(3) \qquad (1.47)
\end{aligned}
$$

because

$$\int_0^\infty dx \, e^{-kx} = \frac{1}{k} \qquad \text{and} \qquad \zeta(z) = \sum_{\nu=1}^\infty \frac{1}{\nu^z} \, .$$

Thus the critical temperature is

$$k_B T_c = \hbar\omega_0 \left(\frac{N}{\zeta(3)}\right)^{1/3} = 0.94 \, \hbar\omega_0 \, N^{1/3} \, . \qquad (1.48)$$

So, for $T < T_c$ we get from the above formulae

$$N = N_0 + \left(\frac{k_B T}{\hbar\omega_0}\right)^3 \zeta(3) \, , \qquad (1.49)$$

hence

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^3 \, . \qquad (1.50)$$

Therefore, we have a situation analogous to the one of free Bosons: $T_c \neq 0$ and the condensate starts forming at $T = T_c$ and, eventually, contains all Bosons at $T = 0$.

Our discussion of condensation of Bosons can summarized as follows: As long as the temperature, $k_B T$, is much larger than the average distance between neighbouring levels, $\Delta\epsilon$, of a system of Bosons, and we are reasonably close to the thermodynamic limit ($N \to \infty$, the size of the system $\to \infty$, the density kept constant), the onset of the critical temperature, T_c, is well defined. However, when $k_B T < \Delta\epsilon$ *there is no critical temperature*: the dependence of the condensate on T is smooth. All this is well illustrated by the system of Bosons confined by a harmonic oscillator potential.