1 Schrödinger's Cat[1]

One of the intepretational problems of QM consits in a fact that the system can be in a superposition of two states $|\phi\rangle$ and $|\psi\rangle$given as

$$\sqrt{\frac{1}{2}}\left(|\phi\rangle + |\psi\rangle\right)$$

even if being in one of these states excludes the aother one. A typical example is a superposition of two states of a cat being alive or dead. While quantum superposition of microscopic states is not particularly strange, as it is essential for quantum interference effectcs, a superposition of macroscopic, *classical* states (like a cat) seems to be paradoxical. There is one very important feature that defines a macroscopic state: it is a state that is by itself a superposition of a large number of single microscopic states. We will show that it is possible to construct a superposition of classical antinomic states, however such superpositions are practically not detectable and very fragile.

1.1 Harmonic osillator - remeinder

Consider one-dimensional harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \tag{1}$$

that we will solve with the help of creation and annihilation operators. It is convenient to define dimensionless operators

$$\hat{\xi} = \sqrt{\frac{m\omega}{\hbar}}\hat{x},\ \hat{\pi} = \frac{1}{\sqrt{m\hbar\omega}}\hat{p}. \tag{2}$$

Then

$$\hat{H} = \frac{1}{2}\hbar\omega\left(\hat{\pi}^2 + \hat{\xi}^2\right) \tag{3}$$

and

$$\hat{a} = \sqrt{\frac{1}{2}}\left(\hat{\xi} + i\hat{\pi}\right),\ \hat{a}^\dagger = \sqrt{\frac{1}{2}}\left(\hat{\xi} - i\hat{\pi}\right) \tag{4}$$

and

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right),\ \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}^\dagger - \hat{a}\right) \tag{5}$$

Note that

$$\left[\hat{\xi}, \hat{\pi}\right] = i,\ \left[\hat{a}, \hat{a}^\dagger\right] = 1. \tag{6}$$

[1] J.-L. Basdevant, L. Dalibard *The Quantum Mechanics Solver*

Recall that

$$\begin{aligned}\hat{a}^\dagger\hat{a}\,|n\rangle &= n\,|n\rangle\,,\\ \hat{a}\,|n\rangle &= \sqrt{n}\,|n-1\rangle\,,\\ \hat{a}^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle\end{aligned} \tag{7}$$

and

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a}\,+\frac{1}{2}\right). \tag{8}$$

In configuration representation $\hat{\pi} = -i\partial/\partial\xi$ and in momentum representation $\hat{\xi} = i\partial/\partial\pi$.

1.2 Coherent states

A good model for a classical state is a *coherent state*, i.e. the normalized eigen state of the annihilation operator $\hat{a}$:

$$|z\rangle = e^{-|z|^2/2}\sum_{n=0}\frac{z^n}{\sqrt{n!}}\,|n\rangle \tag{9}$$

where z is a complex number. Indeed

$$\begin{aligned}\hat{a}\,|z\rangle &= e^{-|z|^2/2}\sum_{n=1}\frac{z^n}{\sqrt{n!}}\sqrt{n}\,|n-1\rangle\\ &= e^{-|z|^2/2}\sum_{n=0}\frac{z^{n+1}}{\sqrt{n!}}\sqrt{n}\,|n\rangle\\ &= z\,|z\rangle\,.\end{aligned} \tag{10}$$

This means

$$\langle z|\,\hat{a}^\dagger = \langle z|\,z^* \tag{11}$$

Let's calculate some properties of the coherent states.

Mean energy:

$$\langle z|\,\hat{H}\,|z\rangle = \hbar\omega\,\langle z|\left(\hat{a}^\dagger\hat{a}\,+\frac{1}{2}\right)|z\rangle = \hbar\omega\,\left(|z|^2+\frac{1}{2}\right), \tag{12}$$

mean position and momentum:

$$\bar{x} = \langle z|\,\hat{x}\,|z\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,(z^*+z)\,,\;\bar{p} = \langle z|\,\hat{p}\,|z\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\,(z^*-z)\,. \tag{13}$$

Mean square deviations:

$$\Delta x^2 = \langle z|\,(\hat{x}-\bar{x})^2\,|z\rangle = \langle z|\,\hat{x}^2 - 2\bar{x}\hat{x} + \bar{x}^2\,|z\rangle = \langle z|\,\hat{x}^2\,|z\rangle - \bar{x}^2. \tag{14}$$

Note that

$$\begin{aligned}\hat{x}^2 &= \frac{\hbar}{2m\omega}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\right) \\ &= \frac{\hbar}{2m\omega}\left(\hat{a}^\dagger\hat{a}^\dagger + 2\hat{a}^\dagger\hat{a} + \hat{a}\hat{a} + 1\right). \qquad (15)\end{aligned}$$

Hence

$$\begin{aligned}\Delta x^2 &= \frac{\hbar}{2m\omega}\left[(z^* + z)^2 + 1 - (z^* + z)^2\right] \\ &= \frac{\hbar}{2m\omega}. \qquad (16)\end{aligned}$$

Similarly

$$\Delta p^2 = \langle z|\,\hat{p}^2\,|z\rangle - p^2 \qquad (17)$$

with

$$\begin{aligned}\hat{p}^2 &= -\frac{m\omega\hbar}{2}\left(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\right) \\ &= -\frac{m\omega\hbar}{2}\left(\hat{a}^\dagger\hat{a}^\dagger - 2\hat{a}^\dagger\hat{a} + \hat{a}\hat{a} - 1\right) \qquad (18)\end{aligned}$$

and

$$\begin{aligned}\Delta p^2 &= -\frac{m\omega\hbar}{2}\left[(z^* - z)^2 - 1 - (z^* - z)^2\right] \\ &= \frac{m\omega\hbar}{2}. \qquad (19)\end{aligned}$$

Note that coherent states for any z saturate uncertainty principle (like the ground state of the harmonic oscillator)

$$\Delta x^2 \Delta p^2 = \frac{\hbar^2}{4}. \qquad (20)$$

To calcuate explicit form of the wave functions we shall use (10):

$$\sqrt{\frac{1}{2}}\left(\xi + \frac{d}{d\xi}\right)\psi_z(\xi) = z\psi_z(\xi). \qquad (21)$$

The solution reads:

$$\psi_z(\xi) = C\exp\left(-\frac{1}{2}(\xi - \sqrt{2}z)^2\right). \qquad (22)$$

Similarly in the momentum space:

$$i\sqrt{\frac{1}{2}}\left(\pi + \frac{d}{d\pi}\right)\tilde{\psi}_z(\pi) = z\tilde{\psi}_z(\pi). \qquad (23)$$

And the solution corresponds to $\psi_z(\xi)$ with $z \to -iz$:

$$\tilde{\psi}_z(\pi) = \tilde{C} \exp\left(-\frac{1}{2}(\pi + i\sqrt{2}z)^2\right). \tag{24}$$

Time dependence of coherent states:

$$\begin{aligned} |z,t\rangle &= e^{-|z|^2/2} \sum_{n=0} \frac{z^n}{\sqrt{n!}} e^{-E_n t/\hbar} |n\rangle \\ &= e^{-|z|^2/2} e^{-i\omega t/2} \sum_{n=0} \frac{z^n}{\sqrt{n!}} \left(e^{-\omega t}\right)^n |n\rangle \\ &= e^{-i\omega t/2} |z(t)\rangle \end{aligned} \tag{25}$$

with

$$z(t) = ze^{-i\omega t}. \tag{26}$$

Assume

$$z = \rho e^{i\varphi} \tag{27}$$

then

$$\begin{aligned} \langle z,t|\,\hat{x}\,|z,t\rangle &= \sqrt{\frac{2\hbar}{m\omega}}\rho\cos(\omega t - \varphi) = x_0 \cos(\omega t - \varphi), \\ \langle z,t|\,\hat{p}\,|z,t\rangle &= -\sqrt{2\hbar m\omega}\rho\sin(\omega t - \varphi) = -p_0 \sin(\omega t - \varphi) \end{aligned} \tag{28}$$

with

$$x_0 = \sqrt{\frac{2\hbar}{m\omega}}\rho,\ p_0 = \sqrt{2\hbar m\omega}\rho. \tag{29}$$

Note that this is motion of a classical oscillator. For semiclassical approximation we shall assume $\rho \gg 1$. Using (16) and (19) we have

$$\frac{\Delta x}{x_0} = \frac{1}{2\rho} \ll 1,\ \frac{\Delta p}{p_0} = \frac{1}{2\rho} \ll 1. \tag{30}$$

Relative uncertainties are time independent and very small for a semiclassical state.

1.3 Construction of a Schrödinger's cat

In time interval $[0,T]$ we switch "perturbation"

$$\hat{W} = \hbar g \left(\hat{a}^\dagger \hat{a}\right)^2. \tag{31}$$

Assume $g \gg \omega$ and $\omega T \ll 1$. This means

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \hat{W} \simeq \hat{W}. \tag{32}$$

Assume initial condition at time $t = 0$:

$$|\psi(0)\rangle = |z\rangle . \tag{33}$$

Since

$$\hat{W}|n\rangle = \hbar g n^2 |n\rangle \tag{34}$$

time dependence takes the following form

$$|\psi(t)\rangle = e^{-|z|^2/2} \sum_{n=0} \frac{z^n}{\sqrt{n!}} e^{-ign^2 t} |n\rangle . \tag{35}$$

Tis is rather complicated time dependence, but it simplifies for some particular values of T.

- $T = 2\pi/g$

  $$e^{-ign^2 T} = 1$$

  and

  $$|\psi(T)\rangle = |z\rangle . \tag{36}$$

- $T = \pi/g$

  $$e^{-ign^2 T} = (-1)^n$$

  since it is 1 for even n and -1 for odd n. Therefore

  $$|\psi(T)\rangle = |-z\rangle . \tag{37}$$

- $T = \pi/2g$

  $$\begin{aligned} e^{-ign^2 T} &= e^{-in^2\pi/2} = \begin{cases} 1 & \text{for} \quad n \text{ - even} \\ -i & \text{for} \quad n \text{ - odd} \end{cases} \\ &= \frac{1}{2}\left[1 - i + (-)^n (1+i)\right] \\ &= \frac{1}{\sqrt{2}}\left(e^{-i\pi/4} + (-)^n e^{i\pi/4}\right) . \end{aligned} \tag{38}$$

  In this case

  $$\begin{aligned} |\psi(T)\rangle &= e^{-|z|^2/2} \frac{1}{\sqrt{2}} \sum_{n=0} \left(e^{-i\pi/4} + (-)^n e^{i\pi/4}\right) \frac{z^n}{\sqrt{n!}} |n\rangle \\ &= \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}|z\rangle + e^{i\pi/4}|-z\rangle\right) . \end{aligned} \tag{39}$$

Note that states $|z\rangle$ and $|-z\rangle$ are classically distinguishable for $z=\rho$ since average positions differ by a sign and for large ρ are therefore antinomic. They are therefore good models for Schrödinger's cat being *live* or *dead*. For $z=i\rho$ mean position is $\bar{x}=0$, however two states $|z\rangle$ and $|-z\rangle$ have opposite velocities.

We shall calculate probability $P(\xi)$ and $P(\pi)$. In configuration space

$$\begin{aligned} P(\xi) &\sim \left|e^{-i\pi/4}\psi_z(\xi)+e^{i\pi/4}\psi_{-z}(\xi)\right|^2 \\ &= \left|\psi_z(\xi)\right|^2+\left|\psi_{-z}(\xi)\right|^2+e^{i\pi/2}\psi_z^*(\xi)\psi_{-z}(\xi)+e^{-i\pi/2}\psi_{-z}^*(\xi)\psi_z(\xi) \end{aligned} \tag{40}$$

where

$$\begin{aligned} \left|\psi_z(\xi)\right|^2 &= |C|^2\exp\left(-\frac{1}{2}(\xi-\sqrt{2}z^*)^2-\frac{1}{2}(\xi-\sqrt{2}z)^2\right) \\ &= |C|^2\exp\left(-\frac{1}{2}(\xi^2-2\sqrt{2}\xi z^*+2z^{*2})-\frac{1}{2}(\xi^2-2\sqrt{2}\xi z+2z^2)\right) \\ &= |C|^2\exp\left(-\xi^2+\sqrt{2}\xi(z^*+z)-(z^{*2}+z^2)\right). \end{aligned} \tag{41}$$

In momentum space $z\to -iz$ and $\xi\to\pi$:

$$\left|\tilde{\psi}_z(\pi)\right|^2=\left|\tilde{C}\right|^2\exp\left(-\pi^2+i\sqrt{2}\pi(z^*-z)+(z^{*2}+z^2)\right) \tag{42}$$

Interference term in configuration space can be obtained from (41) by replacing $z\to -z$:

$$\psi_z^*(\xi)\psi_{-z}(\xi)=|C|^2\exp\left(-\xi^2+\sqrt{2}\xi(z^*-z)-(z^{*2}+z^2)\right)$$

Now we shall use $z=i\rho$:

$$\left|\psi_{\pm i\rho}(\xi)\right|^2=|C|^2\exp\left(-\xi^2+2\rho^2\right) \tag{43}$$

and

$$\psi_z^*(\xi)\psi_{-z}(\xi)=|C|^2\exp\left(-\xi^2+2\rho^2-i2\sqrt{2}\xi\rho\right) \tag{44}$$

Hence

$$\begin{aligned} P(\xi) &\sim \exp\left(-\xi^2+2\rho^2\right)\left[2+\exp\left(-i2\left(\sqrt{2}\xi\rho-\frac{\pi}{4}\right)\right)+\exp\left(i2\left(\sqrt{2}\xi\rho-\frac{\pi}{4}\right)\right)\right] \\ &= 2\exp\left(-\xi^2+2\rho^2\right)\left[1+\cos\left(2\left(\sqrt{2}\xi\rho-\frac{\pi}{4}\right)\right)\right] \\ &= 4\exp\left(-\xi^2+2\rho^2\right)\cos^2\left(\sqrt{2}\xi\rho-\frac{\pi}{4}\right). \end{aligned} \tag{45}$$

In momentum space

$$\begin{aligned} \left|\tilde{\psi}_{i\rho}(\pi)\right|^2 &= \left|\tilde{C}\right|^2\exp\left(-\pi^2+2\sqrt{2}\pi\rho-2\rho^2\right) \\ &= \left|\tilde{C}\right|^2\exp\left(-\left(\pi-\sqrt{2}\rho\right)^2\right), \\ \left|\tilde{\psi}_{-i\rho}(\pi)\right|^2 &= \left|\tilde{C}\right|^2\exp\left(-\left(\pi+\sqrt{2}\rho\right)^2\right). \end{aligned} \tag{46}$$

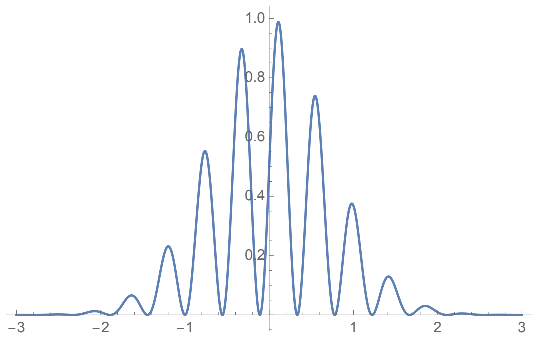

Figure 1: Probability in configuration space.

Interference term

$$\tilde{\psi}_z^*(\pi)\tilde{\psi}_{-z}(\pi) = \left|\tilde{C}\right|^2 \exp\left(-\frac{1}{2}(\pi - \sqrt{2}\rho)^2\right) \exp\left(-\frac{1}{2}(\pi + \sqrt{2}\rho)^2\right) \tag{47}$$

is almost zero because two Gausses have small overlap for large ρ. Therefore

$$P(\pi) \sim \exp\left(-\left(\pi - \sqrt{2}\rho\right)^2\right) + \exp\left(-\left(\pi + \sqrt{2}\rho\right)^2\right). \tag{48}$$

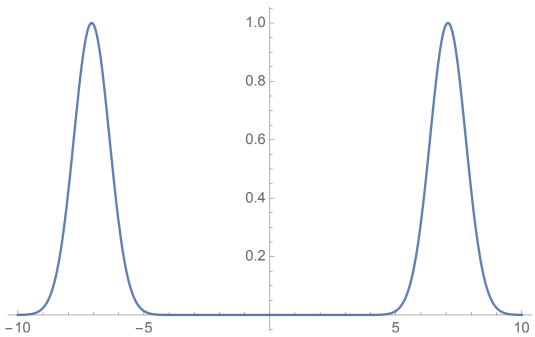

Figure 2: Probability in momentum space.

1.4 Schrödinger's cat vs. statistical superposition

Can one distinguish superposition (39) from a statistical mixture of states $|z\rangle$ and $|-z\rangle$? In order to measure momenta we have to have resolution δp such that

$$\sqrt{m\hbar\omega} \ll \delta p \ll p_0. \tag{49}$$

Consider simple pendulum of $m = 1$ g and 1 m length. Then

$$\omega = \sqrt{\frac{g}{l}} = 3.13\,\frac{1}{\mathrm{s}}. \tag{50}$$

Let's assume that at time $t = 0$ pendulum is 1 μm from equlibrium:

$$x_0 = \sqrt{\frac{2\hbar}{m\omega}}\rho \quad \rightarrow \quad \rho = \sqrt{\frac{m\omega}{2\hbar}}x_0 = \sqrt{\frac{3.13}{2 \times 1.054}10^{34}}\sqrt{\frac{\text{g/s}}{\text{J s}}}\mu\text{m} = 3.85 \times 10^9. \tag{51}$$

Remember that J=kg m^2/s^2 = 10^{15}g μm^2/s^2 and $\hbar = 1.054 \times 10^{-34}$J s. Fom this we have that uncertainty is

$$\frac{\Delta x}{x_0} = \frac{1}{2\rho} \times 10^{-10}. \tag{52}$$

For the momentum distribution

$$\begin{aligned} p_0 &= \sqrt{2\hbar m\omega}\rho = \sqrt{2 \times 1.054 \times 10^{-34\times 10} \times 3.13}\sqrt{10^3\text{g m}^2/\text{s} \times 1/\text{s}} \times 3.85 \times 10^9 \\ &= 3.13 \times 10^{-6}\frac{\text{g m}}{\text{s}}. \end{aligned}$$

This requires spacial resolution better than 1 μm, which is reasonable, given the initial condition. In order to resolve spacial oscillation one needs ξ resolution better than

$$\delta\xi \ll \frac{\pi}{\sqrt{2}\rho} \tag{53}$$

which translates for x

$$\delta x \ll \sqrt{\frac{\hbar}{m\omega}}\frac{\pi}{\sqrt{2}\rho} = \sqrt{\frac{1.054 \times 10^{-34}}{10^{-3}3.13}}\sqrt{\frac{\text{kg m}^2/\text{s}}{\text{kg/s}}}\frac{\pi}{\sqrt{2}3.85 \times 10^9} = 10^{-25} \text{ m}. \tag{54}$$

Such resolution is impossible to attain in practice.

Theoretically, however, a statistical ensemble of states $|z\rangle$ and $|-z\rangle$ would give the same momentum distribution as (39), however a competely different spacial distribution. In the first case the distribution is simply a Gaussian, and in the latter a Gaussian enveloping the oscillations.

1.5 Fragility of a quantum superposition

Assume that the oscillator is in some way coupled with an (non-thermal) environement, whose quantum state will be denoted as $|\chi\rangle$. We shall try to estimate how long the system will stay in a superposition state (39). Let us first consider coupling of a coherent state. Initially at $t = 0$ the sytem is in a state $|\Phi(0)\rangle$

$$|\Phi(0)\rangle = |z(0)\rangle\,|\chi(0)\rangle\,, \tag{55}$$

Assume that time evolution is now modified:

$$z(t) \rightarrow z_\gamma(t) = z(t)e^{-\gamma t} \tag{56}$$

where $z(t)$ corresponds to (26). So in time t the state is now

$$|\Phi(t)\rangle = \left|z(t)e^{-\gamma t}\right\rangle |\chi(t)\rangle\,. \tag{57}$$

This means that the energy of an oscillator part of such a state is now

$$E_{\text{osc}} = \hbar\omega \left(|z|^2 e^{-2\gamma t} + \frac{1}{2} \right). \tag{58}$$

After time much longer than $1/\gamma$ the system goes to a ground state. The energy gained by environement is therefore

$$\Delta E(t) = |z|^2 (1 - e^{-2\gamma t}) \simeq 2\gamma t \, |z|^2 \,, \tag{59}$$

where the last equality holds for short times $2\gamma t \ll 1$. Let us now couple Schrödinger's cat state with the environement

$$|\Phi(t)\rangle = \frac{1}{\sqrt{2}} \left(e^{-i\pi/4} \, |z_\gamma(t)\rangle \left|\chi^{(+)}(t)\right\rangle + e^{i\pi/4} \, |-z_\gamma(t)\rangle \left|\chi^{(-)}(t)\right\rangle \right), \tag{60}$$

where $\left|\chi^{(\pm)}(t)\right\rangle$ are two normalized states of the environement that are a priori different (but not orthogonal). Let's choose again $z = i\rho$ with ρ being large. Then

$$P(x) = \frac{1}{2} \left[\left|\psi_{z_\gamma}(x)\right|^2 + \left|\psi_{-z_\gamma}(x)\right|^2 + 2\mathbf{Re} \left(i\psi^*_{z_\gamma}(x)\psi_{-z_\gamma}(x) \right) \left\langle \chi^{(+)}(t) \,\middle|\, \chi^{(-)}(t) \right\rangle \right], \tag{61}$$

where we assume that

$$\left\langle \chi^{(+)}(t) \,\middle|\, \chi^{(-)}(t) \right\rangle = \eta \in \mathcal{R}, \quad 0 < \eta < 1. \tag{62}$$

Going back to the dimensionless variables we see that the probability distribution in the confifuration space

$$P(\xi) = 2 \exp \left(2(\rho e^{-\gamma t})^2 \right) \exp \left(-\xi^2 \right) \left[1 + \eta \cos \left(2 \left(\sqrt{2}\xi(\rho e^{-\gamma t}) - \frac{\pi}{4} \right) \right) \right] \tag{63}$$

has still the Gaussian envelope, but the oscillatory term is suppressed by η. One can in principle still see the quantum wiggles in a position distribution if η is not too small.

Momentum space distribution does not change much, because the interference term did not contribute. One recovers two peaks centered at $\pm\rho e^{-\gamma t}\sqrt{2m\hbar w}$.

Assume now that the environement is represented by a harmonic oscillator of the same mass and frequency. Assume that initially the environement is in a ground state

$$|\chi\,(0)\rangle = |0\rangle \,.$$

If the coupling between the two oscillators is quadratic (as in $\hat{W}$) we will assume that in the course of time

- $\left|\chi^{(\pm)}(t)\right\rangle$ are coherent states $\left|\chi^{(\pm)}(t)\right\rangle = |\pm y\rangle$
- and for short times $|y|^2 = 2\gamma t \, |z|^2$

Then

$$\eta = \left\langle \chi^{(+)}(t) \left| \chi^{(-)}(t) \right\rangle = e^{-|y|^2} \sum_n \frac{1}{n!} y^{*n} (-y)^n = e^{-2|y|^2} \tag{64}$$

If we want η not too small $|y|^2 < 1$. For short times the energy of the first oscillator

$$E(t) = E(0) - 2\hbar\omega\gamma t \left|z\right|^2 \tag{65}$$

and of the second

$$E'(t) = \hbar\omega \left(2\gamma t \left|z\right|^2 + \frac{1}{2} \right). \tag{66}$$

Total energy is conserved. Once the energy is transferred from the first oscillator to the second, the first oscillator becomes less and less semiclassical. Suppose that $1/2\gamma = 1$ year $= 3 \times 10^7$s, the time to reach $|y|^2 = 1$

$$t = \frac{1}{2\gamma} \frac{1}{\rho^2} = \frac{3 \times 10^7}{(3.85 \times 10^9)^2} \text{s} = 2 \times 10^{-12} \text{s}. \tag{67}$$

To conclude:

- Even for a system protected from the environement the quantum superpositions of macroscopic states are not observable,
- Interaction with environement will very quickly destroy superposition;
- Attempts on small systems with a limitted number of degrees of freedom have been undertaken, but are inconclusive