

Mechanika Kwantowa dla doktorantów  
zestaw 9 na dzień 3.12.2012 poniedziałek  
godz. 10:30, sala 128

1. Przedyskutować zależność od cechowania otrzymanego na poprzednich zajęciach wzoru na propagator cząstki w jednorodnym polu magnetycznym.
2. Poziomy Landaua – dokończenie. Rozważyć ruch cząstki naładowanej w stałym polu magnetycznym  $B$  skierowanym wzdłuż osi  $z$ . W tym celu należy skonstruować hamiltonian korzystając ze znanego uogólnienia z mechaniki klasycznej  $\vec{p} \rightarrow \left(\vec{p} - \frac{e}{c}\vec{A}\right)$  i wybrać  $\vec{A} = \frac{B}{2}(-y, x, 0)$ .

Wskazówka: potraktować człon liniowy w  $B$  jako zaburzenie i wyrazić go przy pomocy operatorów kreacji i anihilacji. Znaleźć poprawkę do energii w pierwszym rzędzie dla trzech pierwszych poziomów  $n = 0, 1, 2$  i spróbować uogólnić dla dowolnego  $n$ .

3. *Instanton determinant.* In this problem we will calculate explicitly the ratio of determinants

$$K' = \frac{\det' \left( -\frac{d^2}{d\tau^2} + V''(\bar{x}(\tau)) \right)}{\det \left( -\frac{d^2}{d\tau^2} + 1 \right)},$$

where a double well potential  $V(x)$  reads:  $V(x) = \kappa(a^2 - x^2)^2$  with  $\kappa = 1/(8a^2)$ . Prime at the determinant means that the zero eigenvalue (zero mode) is not included, by  $\bar{x}(\tau)$  we denote classical trajectory.

- we have shown that the solutions to the eigen equation

$$\left[ -\frac{d^2}{d\tau^2} + V''(\bar{x}(\tau)) \right] y_n(\tau) = \lambda_n y_n(\tau) \quad (1)$$

read

$$y_n(\tau) = \mathcal{N} \left( 3 \tanh^2 \left( \frac{\tau}{2} \right) - 6\alpha \tanh \left( \frac{\tau}{2} \right) + (4\alpha^2 - 1) \right) e^{\alpha\tau}$$

where

$$\alpha = \pm \sqrt{-E_n} = \pm \sqrt{1 - \lambda_n}.$$

- To find contribution from the continuous spectrum we show first that there is no reflection for the particle scattering over the potential  $U(\tau) = -3/2 \cosh^2(\tau/2)$ . To this end find asymptotics for two types of the solutions:  $\alpha = ik$  and  $\alpha = -ik$  in the limit  $\tau \rightarrow \pm\infty$ .
- If there is no reflection then the wave function  $y_k(\tau)$  that asymptotically behaves as  $e^{ik\tau}$  for  $\tau \rightarrow \infty$ , in the limit of  $\tau \rightarrow -\infty$  behaves as  $e^{ik\tau + i\delta_k}$ , where  $\delta_k$  is a phase shift. Show that

$$e^{i\delta_k} = \frac{1 + ik}{1 - ik} \frac{1 + 2ik}{1 - 2ik}.$$

Identical argument applies to the wave function, that asymptotically behaves as  $e^{-ik\tau}$ .

- Close the system in a box  $-T/2 < \tau < T/2$ . Then the wave function inside the box is a superposition of two linearly independent solutions

$$y_n(\tau) = Ay_{\alpha=ik}(\tau) + By_{\alpha=-ik}(\tau)$$

which vanishes at the boundaries

$$y_n(\pm T/2) = 0. \quad (2)$$

If the box is large, it is enough to use asymptotic forms of  $y_{\alpha=\pm ik}(\tau)$ . Show that condition (2) leads to

$$Tk - \delta_k = \pi n.$$

Let's denote solution to this equation by  $\tilde{k}_n$ . Similarly, show that for the Euclidean harmonic oscillator analogous solutions read  $k_n = \pi n/T$ .

- The contribution to  $K'$  coming from the continuous spectrum,  $K_{cont}$ , reads:

$$K_{cont} = \frac{\prod \tilde{\lambda}_n}{\prod \lambda_n} = \prod_{n=1}^{\infty} \frac{1 + \tilde{k}_n^2}{1 + k_n^2} = \exp\left(\sum_n \ln \frac{1 + \tilde{k}_n^2}{1 + k_n^2}\right) \approx \exp\left(\sum_n \frac{2k_n(\tilde{k}_n - k_n)}{1 + k_n^2}\right).$$

- To calculate last sum under the exponent go to the continuum limit  $T \rightarrow \infty$  and convert the sum into the integral:

$$\dots = \exp\left(\frac{1}{\pi} \int_0^{\infty} dk \frac{2\delta_k k}{1 + k^2}\right) = \frac{1}{9}.$$

Last equality can be obtained by integration by parts and the explicit form of  $\delta_k$ . Full result for  $K'$  is obtained by multiplying  $K_{cont}$  by a non-zero  $\lambda$  value from the discrete part.

- Literature:

S. Coleman, *Aspects of Symmetry*, Cambridge University Press (1988), Section 7, Appendix 1.

A.I. Vainshtein, V.I. Zakharov, V.A. Novikov and M.A. Shifman, *ABC of Instantons*, Sov. Phys. Usp. **24**, 195 (1982) [Usp. Fiz. Nauk **136**, 553 (1982)].

T. Schafer and E.V. Shuryak, *Instantons in QCD*, Rev. Mod. Phys. **70** (1998) 323 [arXiv:hep-ph/9610451].