Mechanika Kwantowa dla doktorantów

zestaw 24 – 18.4.2012

1. Suppose we have solved the radial part of the Schrödinger equation for some central potential $V(r)$ of a finite range r_0. This solution for positive energy $E = \hbar^2 k^2/2m > 0$ can be decomposed in terms of partial waves

$$\psi(\vec{r}) = \sum_l (2l+1) i^l A_l(r) P_l(\cos\vartheta), \tag{1}$$

where $A_l(r)$ are in principle known functions of r (why we decompose this solution into Legendre polynomials rather than in spherical harmonics Y_l^m?). Show that the phase shift is given as

$$\tan\delta_l = \frac{kR\, j_l'(kR) - \beta_l\, j_l(kR)}{kR\, y_l'(kR) - \beta_l\, y_l(kR)}$$

where j_l and y_l are spherical Bessel functions.

HINT

For the scattering problem the wave function should have the following asymptotic form:

$$\psi \to e^{ikz} + f(\theta)\frac{e^{ikr}}{r}.$$

Decomposing (prove it!)

$$e^{ikz} = \sum_l (2l+1) i^l j_l(kr) P_l(\cos\theta)$$

and

$$f(\theta) = \sum_l (2l+1) f_l(\theta) P_l(\cos\theta)$$

one can find asymptotic form of the wave function applying known asymptotic forms of the Bessel functions. Denote a coefficient in front of the outgoing spherical wave e^{ikr} as

$$S_l = e^{2i\delta_l}.$$

This is a definition of the phase shift. Find relation befween f_l and S_l.

Since we know A_l in (1) we can decompose it in terms of Hankel functions (which is the same as decomposing A_l in terms of a Bessel functions. Bessel functions are like sin and cos, while Hankel functions are like exponents):

$$A_l = c_l^{(+)} h_l^{(+)} + c_l^{(-)} h_l^{(-)}$$

where

$$h_l^{(\pm)} = j_l \pm i y_l.$$

Find asymptotics of Hankel functions.

At $r = R \gg r_0$ we can glue ψ with its asymptotic form. This allows to find $c_l^{(\pm)}$. Going back to the decomposition in terms of Bessel functions show that

$$A_l(r) = e^{i\delta_l(k)} \left[j_l(kr) \cos\delta_l(k) - y_l(kr) \sin\delta_l(k) \right]. \tag{2}$$

Since A_l is known, eq. (2) is in fact an equation for δ_l. In practice (2) is difficult to solve, therefore one applies a trick constructing a quantity β

$$\beta_l = \left. \frac{r}{A_l} \frac{dA_l}{dr} \right|_{r=R}. \tag{3}$$

Note that β is known. Solve (3) for $\tan\delta_l$:

$$\tan\delta_l = \frac{kR\, j_l'(kr) - \beta_l\, j_l(kr)}{kR\, y_l'(kr) - \beta_l\, y_l(kr)}.$$

2. The situation dramatically simplifies for an infinte "hard ball":

$$V(r) = \begin{cases} 0 & \text{dla} \quad R < r \\ \infty & \text{dla} \quad r < R \end{cases}$$

since then the phase shifts can be calculated from the conditon $A_l(R) = 0$ (2). Find low energy behaviour of δ_l. Calculate the cross-section for the lowest partial wave $l = 0$. As you will see the cross-section is not geometrical i.e $\sigma \neq \pi R^2$.

3. Sum up higher partial waves

$$\sigma_{\rm tot} = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2\delta_l(k)$$

up to a maximal clasically allowed $l_{\max} \sim kR$. To this end use

$$\sin^2\delta_l(k) = \frac{\tan^2\delta_l(k)}{1 + \tan^2\delta_l(k)}$$

and the formula for $\tan\delta_l(k)$ in terms of spherical Bessel functions. Then use asymptotic form of Bessel functions. The resulting cross-section is still not geometrical ($\sigma_{\rm tot} = 2\pi R^2$). Try to interpret this result (Sakurai, Advanced Quantum Mechanics, Chapt.7.6).