Mechanika Kwantowa dla doktorantów
zestaw 10 na dzień 4.1.2012 środa

1. Consider Euclidean motion (in an inverted potential) of a given energy $E < 0$, leading from $x_1 \to x_2$ ($x_1 < x_2$) in time T. As a potential take $V(x) = \kappa(x^2 - a^2)^2$ with $\kappa = 1/8a^2$. For one instanton-like motion (without turning) it is clear that as $T \to \infty$ then $x_1 \to -a$, $x_2 \to a$ and $E \to 0$. Show that in this limit

$$E = -8a^2 e^{-T}.$$

HINT. Use classical formula for T. In the limit $E \to 0$ the integral will contain a singular part that can be divided into a finite and still singular part by subtracting and adding $((x-a)^2 + 2E)^{-1/2}$. In the finite part one can immediately set $E = 0$. The other part can be calculated exactly for finite, but small E.

2. *Instanton determinant.* In this problem we will calculate explicitly the ratio of determinants

$$K' = \frac{\det'\left(-\frac{d^2}{d\tau^2} + V''(\bar{x}(\tau))\right)}{\det\left(-\frac{d^2}{d\tau^2} + 1\right)},$$

where a double well potential $V(x)$ reads: $V(x) = \kappa(a^2 - x^2)^2$ with $\kappa = 1/(8a^2)$. Prime at the determinant means that the zero eigenvalue (zero mode) is not included, by $\bar{x}(\tau)$ we denote classical trajectory.

- The eigenequation for quantum fluctuations around the classical trajectory (with $\tau_1 = 0$, where τ_1 is the time when the classical trajectory passes through zero):

$$\left[-\frac{d^2}{d\tau^2} + V''(\bar{x}(\tau))\right] y_n(\tau) = \lambda_n y_n(\tau) \qquad (1)$$

corresponds to the Schrödinger equation for a potential $U(\tau) = -3/(2\cosh^2(\tau/2))$ and energy $E_n = \lambda_n - 1$, which is discussed in the "Quantum Mechanics" of Landau and Lifischitz (probl. 5 page. 81 and probl. 4 page 88, Polish edition PWN 1979).

Transform equation (1) into a hypergeometric equation for function w_n defined below:

$$y_n(\tau) = e^{\alpha\tau} w_n(\tau),$$

where

$$\alpha = \pm\sqrt{-E_n}, \qquad E_n = \lambda_n - 1.$$

Show that the solution reads:

$$y_n(\tau) = \mathcal{N}\left(3\tanh^2\left(\frac{\tau}{2}\right) - 6\alpha\tanh\left(\frac{\tau}{2}\right) + \left(4\alpha^2 - 1\right)\right) e^{\alpha\tau}.$$

- Find spectrum of the bound states ($E < 0$) for (1). Conditions that solutions vanish at $\tau = \pm\infty$ give quantization of α.
- To find contribution from the continous spectrum we show first that there is no reflection for the particles scattering over the potential $U(\tau)$. To this end find asymptotics for two types of the solutions: $\alpha = ik$ and $\alpha = -ik$ in the limit $\tau \to \pm\infty$.
- If there is no reflection then the wave function $y_k(\tau)$ that asymptotically behaves as $e^{ik\tau}$ for $\tau \to \infty$, in the limit of $\tau \to -\infty$ behaves as $e^{ik\tau + i\delta_k}$, where δ_k is a phase shift. Show that

$$e^{i\delta_k} = \frac{1+ik}{1-ik}\frac{1+2ik}{1-2ik}.$$

  Identical argument applies to the wave function, that asymptotically behaves as $e^{-ik\tau}$.
- Close the system in a box $-T/2 < \tau < T/2$. Then the wave function inside the box is a superposition of two linearly independent solutions

$$y_n(\tau) = Ay_{\alpha=ik}(\tau) + By_{\alpha=-ik}(\tau)$$

  which vanishes at the boundaries

$$y_n(\pm T/2) = 0. \tag{2}$$

  If the box is large, it is enough to use asymptotic forms of $y_{\alpha=\pm ik}(\tau)$. Show that condition (2) leads to

$$Tk - \delta_k = \pi n.$$

  Let's denote solution to this equation by $\tilde{k}_n$. Similarly, for the Euclidean harmonic oscillator analogous solutions read $k_n = \pi n/T$.
- The contribution to K' coming from the continuous spectrum, K_{cont}, reads:

$$K_{cont} = \frac{\prod \tilde{\lambda}_n}{\prod \lambda_n} = \prod_{n=1}^{\infty} \frac{1+\tilde{k}_n^2}{1+k_n^2} = \exp\left(\sum_n \ln\frac{1+\tilde{k}_n^2}{1+k_n^2}\right) \approx \exp\left(\sum_n \frac{2k_n(\tilde{k}_n - k_n)}{1+k_n^2}\right).$$

- To calculate last sum under exponent go to the continuum limit $T \to \infty$ and convert the sum into the integral:

$$\ldots = \exp\left(\frac{1}{\pi}\int_0^{\infty} dk \frac{2\delta_k\, k}{1+k^2}\right) = \frac{1}{9}.$$

  Last equality can be obtained by integration by parts and the explicit form of δ_k. Full result for K' is obtained by multiplying K_{cont} by a non-zero λ value from the discrete part.

- Literature:

  S. Coleman, *Aspects of Symmetry*, Cambridge University Press (1988), Section 7, Appendix 1.

  A.I. Vainshtein, V.I. Zakharov, V.A. Novikov and M.A. Shifman, *ABC of Instantons*, Sov. Phys. Usp. **24**, 195 (1982) [Usp. Fiz. Nauk **136**, 553 (1982)].

  T. Schafer and E.V. Shuryak, *Instantons in QCD*, Rev. Mod. Phys. **70** (1998) 323 [arXiv:hep-ph/9610451].