Mechanika Kwantowa dla doktorantów
zestaw 8.5 na dzień 21.12.2011 środa

1. Prove the following identities:
$$\binom{N+1}{\mu} = \binom{N}{\mu} + \binom{N}{\mu-1},$$
$$\sum_{\mu''\le\mu'\le\mu} \binom{N}{\mu-\mu'}\binom{N'}{\mu'-\mu''} = \binom{N+N'}{\mu-\mu''}.$$

2. Show that for large N
$$\binom{N}{\mu}\left(\frac{1}{2}\right)^N \approx \sqrt{\frac{2}{\pi N}}e^{-\frac{j^2}{2N}}.$$

3. For $V(x) = \kappa(a^2 - x^2)^2$ with $\kappa = 1/8a^2$
   - find classical trajectory in Euclidean time (i.e. in reversed potential $-V(x)$) leading from $-a$ to a (instanton) or from a to $-a$ (anty-instanton),
   - Calculate the classical action corresponding to such a trajectory. Use the fact that the total energy is zero. Calculate quantum propagator K.
   - Show that the eigenequation for quantum fluctuations around the classical trajectory (with $\tau_1 = 0$, where τ_1 is the time when the classical trajectory passes through zero):
   $$\left[-\frac{d^2}{d\tau^2} + V''(\overline{x}(\tau))\right] y_n(\tau) = \lambda_n y_n(\tau) \tag{1}$$
   corresponds to the Schrödinger equation for a potential $1/\cosh^2(\tau/2)$ and energy $E_n = \lambda_n - 1$, which is discussed in the "Quantum Mechanics" of Landau and Lifischitz (probl. 5 page. 81 and probl. 4 page 88, Polish edition PWN 1979).

4. Consider Euclidean motion (in an inverted potential) of a given energy $E < 0$, leading from $x_1 \to x_2$ ($x_1 < x_2$) in time T. As a potential take $V(x) = \kappa(x^2 - a^2)^2$ with $\kappa = 1/8a^2$. For one instanton-like motion (without turning) it is clear that as $T \to \infty$ then $x_1 \to -a$, $x_2 \to a$ and $E \to 0$. Show that in this limit
$$E = -8a^2e^{-T}.$$

   HINT. Use classical formula for T. In the limit $E \to 0$ the integral will contain a singular part that can be divided into a finite and still singular part by subtracting and adding $((x-a)^2 + 2E)^{-1/2}$. In the finite part one can immediately set $E = 0$. The other part can be calculated exactly for finite, but small E.