Elementary Particle Physics: theory and experiments

Theory:

Symmetries and Quark Model

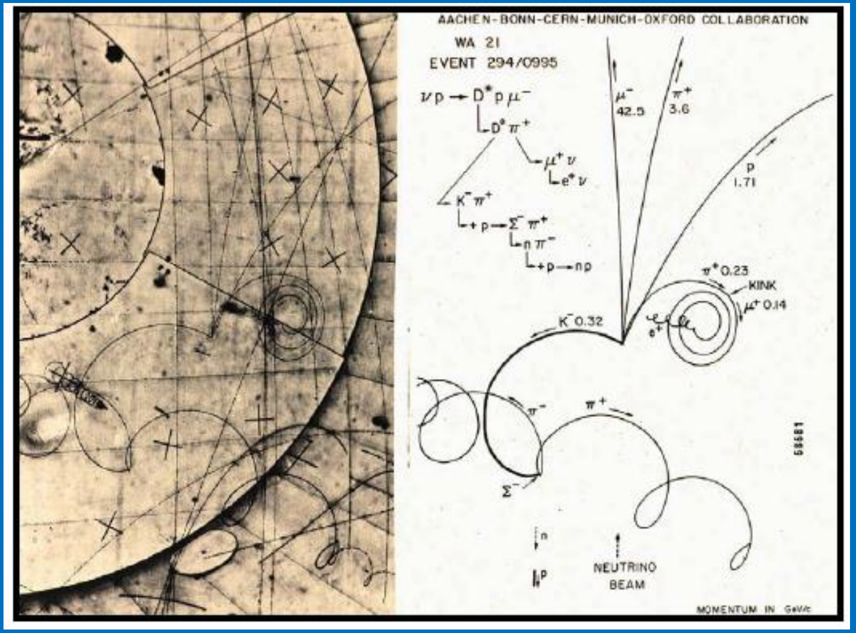
Quantum Chromodynamics

The weak interaction and V-A

Slides taken from M. A. Thomson lectures at Cambridge University in 2011

Prof. dr hab. Elżbieta Richter-Wąs

Symmetries and Quark Model



Symmetries and Conservation Laws

★ Suppose physics is invariant under the transformation

$$\psi \rightarrow \psi' = \hat{U}\psi$$

e.g. rotation of the coordinate axes

• To conserve probability normalisation require

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\hat{U}\psi|\hat{U}\psi\rangle = \langle\psi|\hat{U}^{\dagger}\hat{U}|\psi\rangle$$

$$\Rightarrow \quad \boxed{\hat{U}^{\dagger}\hat{U} = 1}$$

i.e. $\hat{U}$ has to be unitary

• For physical predictions to be unchanged by the symmetry transformation, also require all QM matrix elements unchanged

$$\boxed{\langle\psi|\hat{H}|\psi\rangle = \langle\psi'|\hat{H}|\psi'\rangle} = \langle\psi|\hat{U}^{\dagger}\hat{H}\hat{U}|\psi\rangle$$

i.e. require $\quad \hat{U}^{\dagger}\hat{H}\hat{U} = \hat{H}$

$\times\hat{U} \quad \hat{U}\hat{U}^{\dagger}\hat{H}\hat{U} = \hat{U}\hat{H} \quad \Rightarrow \quad \hat{H}\hat{U} = \hat{U}\hat{H}$

therefore $\quad \boxed{[\hat{H},\hat{U}] = 0}$

$\hat{U}$ commutes with the Hamiltonian

★ Now consider the infinitesimal transformation (ε small)

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

($\hat{G}$ is called the generator of the transformation)

- For $\hat{U}$ to be unitary

$$\hat{U}\hat{U}^{\dagger} = (1 + i\varepsilon\hat{G})(1 - i\varepsilon\hat{G}^{\dagger}) = 1 + i\varepsilon(\hat{G} - \hat{G}^{\dagger}) + O(\varepsilon^2)$$

neglecting terms in ε^2 $\quad UU^{\dagger} = 1$ ➡ $\boxed{\hat{G} = \hat{G}^{\dagger}}$

i.e. $\hat{G}$ is Hermitian and therefore corresponds to an observable quantity G !

- Furthermore, $[\hat{H}, \hat{U}] = 0 \quad \Rightarrow [\hat{H}, 1 + i\varepsilon\,\hat{G}] = 0 \quad \Rightarrow [\hat{H}, \hat{G}] = 0$

But from QM $\quad \dfrac{\mathrm{d}}{\mathrm{d}t}\langle\hat{G}\rangle = i\langle[\hat{H}, \hat{G}]\rangle = 0$

i.e. G is a conserved quantity.

Symmetry ⟷ Conservation Law

★ **For each symmetry of nature have an observable conserved quantity**

Example: Infinitesimal spatial translation $x \rightarrow x + \varepsilon$

i.e. expect physics to be invariant under $\psi(x) \rightarrow \psi' = \psi(x + \varepsilon)$

$$\psi'(x) = \psi(x + \varepsilon) = \psi(x) + \frac{\partial\psi}{\partial x}\varepsilon = \left(1 + \varepsilon\frac{\partial}{\partial x}\right)\psi(x)$$

but $\quad \hat{p}_x = -i\frac{\partial}{\partial x}$ ➡ $\psi'(x) = (1 + i\varepsilon\hat{p}_x)\psi(x)$

The generator of the symmetry transformation is $\hat{p}_x$ ➡ p_x is conserved

- **Translational invariance** of physics implies **momentum conservation !**

- In general the symmetry operation may depend on more than one parameter

$$\hat{U} = 1 + i\vec{\varepsilon}.\vec{G}$$

For example for an infinitesimal 3D linear translation : $\vec{r} \rightarrow \vec{r} + \vec{\varepsilon}$

$$\longrightarrow \quad \hat{U} = 1 + i\vec{\varepsilon}.\vec{p} \qquad \vec{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$$

- So far have only considered an infinitesimal transformation, however a finite transformation can be expressed as a series of infinitesimal transformations

$$\hat{U}(\vec{\alpha}) = \lim_{n \to \infty} \left(1 + i\frac{\vec{\alpha}}{n}.\vec{G}\right)^n = e^{i\vec{\alpha}.\vec{G}}$$

Example: Finite spatial translation in 1D: $x \rightarrow x + x_0$ with $\hat{U}(x_0) = e^{ix_0\hat{p}_x}$

$$\begin{aligned} \psi'(x) = \psi(x + x_0) &= \hat{U}\psi(x) = \exp\left(x_0\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x) \qquad \left(p_x = -i\frac{\partial}{\partial x}\right) \\ &= \left(1 + x_0\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x_0^2}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \ldots\right)\psi(x) \\ &= \psi(x) + x_0\frac{\mathrm{d}\psi}{\mathrm{d}x} + \frac{x_0^2}{2}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \ldots \end{aligned}$$

i.e. obtain the expected Taylor expansion

Symmetries in Particle Physics: Isospin

- The proton and neutron have very similar masses and the nuclear force is found to be approximately charge-independent, i.e.

$$V_{pp} \approx V_{np} \approx V_{nn}$$

- To reflect this symmetry, Heisenberg (1932) proposed that if you could "switch off" the electric charge of the proton

There would be no way to distinguish between a proton and neutron

- Proposed that the neutron and proton should be considered as two states of a single entity; the **nucleon**

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

★ Analogous to the spin-up/spin-down states of a spin-½ particle

ISOSPIN

★ Expect physics to be invariant under rotations in this space

- The neutron and proton form an isospin doublet with total isospin $I = ½$ and third component $I_3 = \pm ½$

Flavour Symmetry of Strong Interaction

We can extend this idea to the quarks:

★ Assume the strong interaction treats all quark flavours equally (it does)

- **Because $m_u \approx m_d$:**

  The strong interaction possesses an approximate flavour symmetry i.e. from the point of view of the strong interaction nothing changes if all up quarks are replaced by down quarks and *vice versa*.

- Choose the basis

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Express the invariance of the strong interaction under $u \leftrightarrow d$ as invariance under "rotations" in an abstract isospin space

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

The 2x2 unitary matrix depends on 4 complex numbers, i.e. 8 real parameters
But there are four constraints from $\hat{U}^{\dagger}\hat{U} = 1$

➡ 8 – 4 = **4 independent matrices**

- In the language of group theory the four matrices form the U(2) group

- One of the matrices corresponds to multiplying by a phase factor

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}$$

not a flavour transformation and of no relevance here.

- The remaining three matrices form an SU(2) group (special unitary) with $\det U = 1$
- For an infinitesimal transformation, in terms of the Hermitian generators $\hat{G}$

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

- $\det U = 1 \quad \Rightarrow \quad Tr(\hat{G}) = 0$
- A linearly independent choice for $\hat{G}$ are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The proposed flavour symmetry of the strong interaction has the same transformation properties as SPIN !
- Define ISOSPIN: $\vec{T} = \frac{1}{2}\vec{\sigma} \qquad \hat{U} = e^{i\vec{\alpha}.\vec{T}}$
- Check this works, for an infinitesimal transformation

$$\hat{U} = 1 + \frac{1}{2}i\vec{\varepsilon}.\vec{\sigma} \quad = \quad 1 + \frac{i}{2}(\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3) = \begin{pmatrix} 1 + \frac{1}{2}i\varepsilon_3 & \frac{1}{2}i(\varepsilon_1 - i\varepsilon_2) \\ \frac{1}{2}i(\varepsilon_1 + i\varepsilon_2) & 1 - \frac{1}{2}i\varepsilon_3 \end{pmatrix}$$

Which is, as required, unitary and has unit determinant

$$U^\dagger U = I + O(\varepsilon^2) \qquad \det U = 1 + O(\varepsilon^2)$$

Properties of Isospin

- Isospin has the exactly the same properties as spin

$$[T_1, T_2] = iT_3 \quad [T_2, T_3] = iT_1 \quad [T_3, T_1] = iT_2$$

$$[T^2, T_3] = 0 \qquad T^2 = T_1^2 + T_2^2 + T_3^2$$

As in the case of spin, have three non-commuting operators, T_1, T_2, T_3 and even though all three correspond to observables, can't know them simultaneously. So label states in terms of total isospin I and the third component of isospin I_3

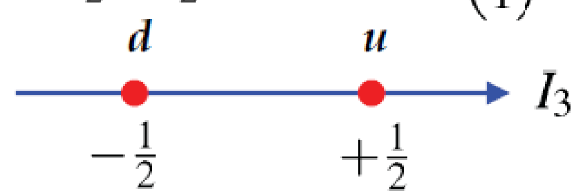
NOTE: isospin has nothing to do with spin – just the same mathematics

- The eigenstates are exact analogues of the eigenstates of ordinary angular momentum $|s, m\rangle \rightarrow |I, I_3\rangle$

  with $T^2|I, I_3\rangle = I(I+1)|I, I_3\rangle \qquad T_3|I, I_3\rangle = I_3|I, I_3\rangle$

- In terms of isospin:

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$



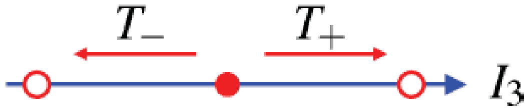
$$I = \tfrac{1}{2}, \quad I_3 = \pm\tfrac{1}{2}$$

- In general $I_3 = \frac{1}{2}(N_u - N_d)$

- Can define isospin ladder operators – analogous to spin ladder operators

$$T_- \equiv T_1 - iT_2$$

u → d



$$T_+ \equiv T_1 + iT_2$$

d → u

$$T_+|I,I_3\rangle = \sqrt{I(I+1)-I_3(I_3+1)}|I,I_3+1\rangle$$

$$T_-|I,I_3\rangle = \sqrt{I(I+1)-I_3(I_3-1)}|I,I_3-1\rangle$$

Step up/down in I_3 until reach end of multiplet $T_+|I,+I\rangle = 0 \quad T_-|I,-I\rangle = 0$

$$T_+u = 0 \qquad T_+d = u \qquad T_-u = d \qquad T_-d = 0$$

- Ladder operators turn $u \to d$ and $d \to u$

★ Combination of isospin: **e.g.** what is the isospin of a system of two **d** quarks, is exactly analogous to combination of spin (i.e. angular momentum)

$$|I^{(1)}, I_3^{(1)}\rangle|I^{(2)}, I_3^{(2)}\rangle \to |I, I_3\rangle$$

- I_3 additive : $I_3 = I_3^{(1)} + I_3^{(2)}$
- I in integer steps from $|I^{(1)} - I^{(2)}|$ to $|I^{(1)} + I^{(2)}|$

★ Assumed symmetry of Strong Interaction under isospin transformations implies the existence of conserved quantites

- In strong interactions I_3 and I are conserved, analogous to conservation of J_z and J for angular momentum

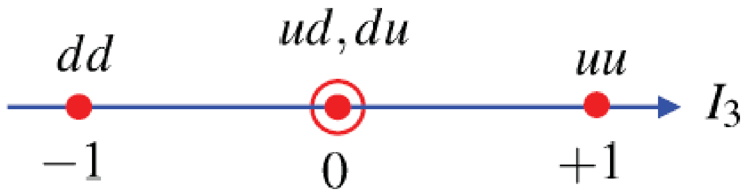
Combining Quarks

Goal: derive proton wave-function

- First combine two quarks, then combine the third
- Use requirement that **fermion** wave-functions are **anti-symmetric**

Isospin starts to become useful in defining states of more than one quark.

e.g. two quarks, here we have four possible combinations:



Note: ⊙ represents two states with the same value of I_3

- We can immediately identify the extremes (I_3 additive)

$$uu \equiv |\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = |1,+1\rangle \qquad dd \equiv |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle = |1,-1\rangle$$

To obtain the $|1,0\rangle$ state use ladder operators

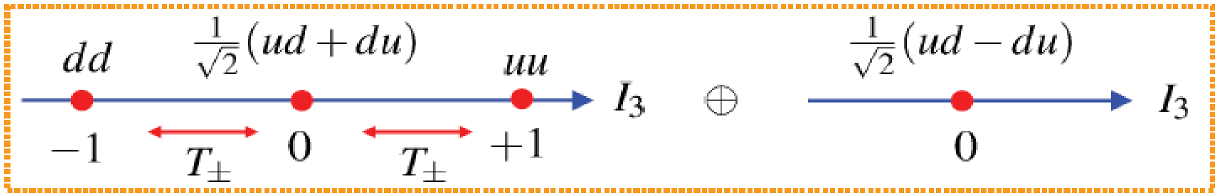
$$T_-|1,+1\rangle = \sqrt{2}|1,0\rangle = T_-(uu) = ud + du$$

$$\Rightarrow \quad |1,0\rangle = \frac{1}{\sqrt{2}}(ud + du)$$

The final state, $|0,0\rangle$, can be found from orthogonality with $|1,0\rangle$

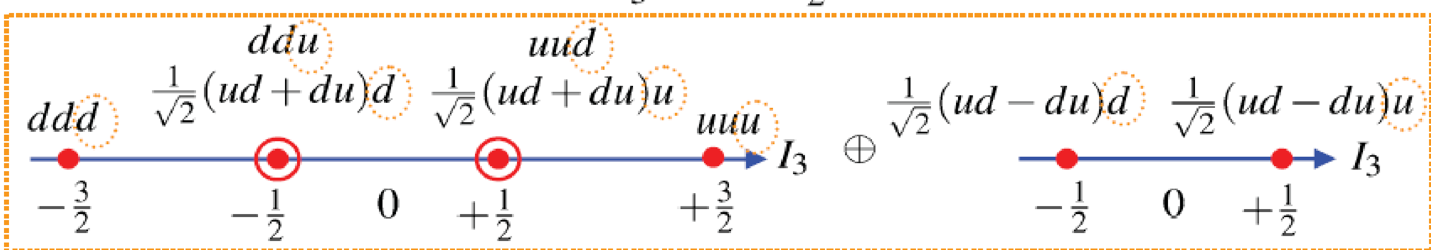
$$\Rightarrow \quad |0,0\rangle = \frac{1}{\sqrt{2}}(ud - du)$$

- From four possible combinations of isospin doublets obtain a **triplet** of **isospin 1** states and a **singlet isospin 0** state $2 \otimes 2 = 3 \oplus 1$



- Can move around <u>within</u> multiplets using ladder operators
- note, as anticipated $I_3 = \frac{1}{2}(N_u - N_d)$
- States with different **total isospin** are physically different – the **isospin 1 triplet** is symmetric under interchange of quarks 1 and 2 whereas **singlet** is anti-symmetric

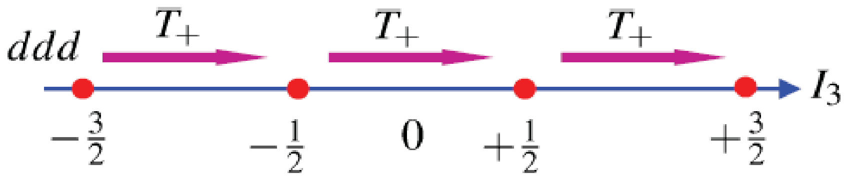
★ Now add an additional up or down quark. From each of the above 4 states get two new isospin states with $I_3' = I_3 \pm \frac{1}{2}$



6 **2**

- Use ladder operators and orthogonality to group the 6 states into isospin multiplets, e.g. to obtain the $I = \frac{3}{2}$ states, step up from ddd

★Derive the $I=\frac{3}{2}$ states from $ddd \equiv |\frac{3}{2},-\frac{3}{2}\rangle$



$$
\begin{aligned}
T_+|\tfrac{3}{2},-\tfrac{3}{2}\rangle &= T_+(ddd) = (T_+d)dd + d(T_+d)d + dd(T_+)d \\
\sqrt{3}|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= udd + dud + ddu \\
|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(udd + dud + ddu) \\
T_+|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}T_+(udd + dud + ddu) \\
2|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uud + udu + uud + duu + udu + duu) \\
|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uud + udu + duu) \\
T_+|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}T_+(uud + udu + duu) \\
\sqrt{3}|\tfrac{3}{2},+\tfrac{3}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uuu + uuu + uuu) \\
|\tfrac{3}{2},+\tfrac{3}{2}\rangle &= uuu
\end{aligned}
$$

★ From the **6** states on previous page, use orthoganality to find $|\frac{1}{2},\pm\frac{1}{2}\rangle$ states

★ The **2** states on the previous page give another $|\frac{1}{2},\pm\frac{1}{2}\rangle$ doublet

★ The eight states uuu, uud, udu, udd, duu, dud, ddu, ddd are grouped into an isospin quadruplet and two isospin doublets

$$2 \otimes 2 \otimes 2 = 2 \otimes (3 \oplus 1) = (2 \otimes 3) \oplus (2 \otimes 1) = 4 \oplus 2 \oplus 2$$

- Different multiplets have different symmetry properties

$$|\tfrac{3}{2},+\tfrac{3}{2}\rangle = uuu$$
$$|\tfrac{3}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(uud+udu+duu)$$
$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(ddu+dud+udd)$$
$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = ddd$$

S — A quadruplet of states which are symmetric under the interchange of <u>any</u> two quarks

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = -\tfrac{1}{\sqrt{6}}(2ddu-udd-dud)$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{6}}(2uud-udu-duu)$$

M_S — Mixed symmetry. Symmetric for 1 ⟷ 2

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(udd-dud)$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(udu-duu)$$

M_A — Mixed symmetry. Anti-symmetric for 1 ⟷ 2

- Mixed symmetry states have no definite symmetry under interchange of quarks $1 \leftrightarrow 3$ etc.

Combining Spin

- Can apply exactly the same mathematics to determine the possible spin wave-functions for a combination of 3 spin-half particles

$$\left|\tfrac{3}{2},+\tfrac{3}{2}\right\rangle = \uparrow\uparrow\uparrow$$
$$\left|\tfrac{3}{2},+\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$$
$$\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow)$$
$$\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle = \downarrow\downarrow\downarrow$$

S — A quadruplet of states which are symmetric under the interchange of any two quarks

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = -\tfrac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow - \uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$
$$\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

M_S — Mixed symmetry. Symmetric for 1 ⟷ 2

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$
$$\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

M_A — Mixed symmetry. Anti-symmetric for 1 ⟷ 2

- Can now form total wave-functions for combination of three quarks

Baryon wave functions (ud)

★ Quarks are fermions so require that the total wave-function is <u>anti-symmetric</u> under the interchange of any two quarks

★ the total wave-function can be expressed in terms of:

$$\psi = \phi_{\text{flavour}}\chi_{\text{spin}}\xi_{\text{colour}}\eta_{\text{space}}$$

★ The colour wave-function for all bound qqq states is <u>anti-symmetric</u> (see handout 8)

- Here we will only consider the lowest mass, **ground state**, baryons where there is no internal orbital angular momentum.
- For **L=0** the spatial wave-function is <u>symmetric</u> $(-1)^L$.

➡ $\xi_{\text{colour}}\eta_{\text{space}}$ anti-symmetric

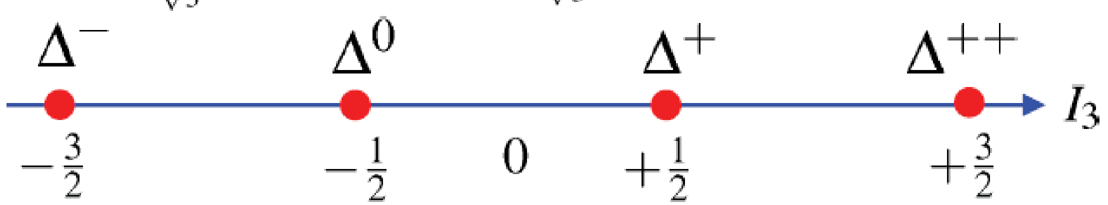
➡ $\phi_{\text{flavour}}\chi_{\text{spin}}$ symmetric

} Overall anti-symmetric

★ <u>Two ways</u> to form a totally symmetric wave-function from spin and isospin states:

❶ combine totally symmetric spin and isospin wave-functions $\phi(S)\chi(S)$

ddd $\quad \frac{1}{\sqrt{3}}(ddu+dud+udd)$ $\quad \frac{1}{\sqrt{3}}(uud+udu+duu)$ $\quad uuu$



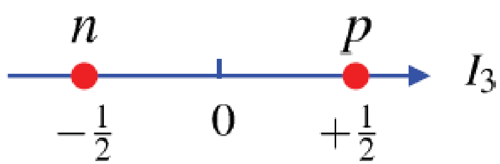
Spin 3/2
Isospin 3/2

❷ combine mixed symmetry spin and mixed symmetry isospin states

- Both $\phi(M_S)\chi(M_S)$ and $\phi(M_A)\chi(M_A)$ are sym. under inter-change of quarks $1 \leftrightarrow 2$
- Not sufficient, these combinations have no definite symmetry under $1 \leftrightarrow 3, \ldots$
- However, it is not difficult to show that the (normalised) linear combination:

$$\frac{1}{\sqrt{2}}\phi(M_S)\chi(M_S) + \frac{1}{\sqrt{2}}\phi(M_A)\chi(M_A)$$

is totally symmetric (i.e. symmetric under $1 \leftrightarrow 2;\ \ 1 \leftrightarrow 3;\ \ 2 \leftrightarrow 3$)



Spin 1/2
Isospin 1/2

- The spin-up proton wave-function is therefore:

$$|p\uparrow\rangle = \frac{1}{6\sqrt{2}}(2uud - udu - duu)(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + \frac{1}{2\sqrt{2}}(udu - duu)(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

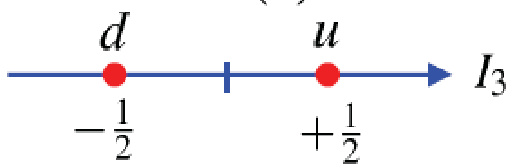
$$\Rightarrow \quad |p\uparrow\rangle = \frac{1}{\sqrt{18}}\big(2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow + 2u\uparrow d\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow - u\downarrow d\uparrow u\uparrow + 2d\downarrow u\uparrow u\uparrow - d\uparrow u\downarrow u\uparrow - d\uparrow u\uparrow u\downarrow\big)$$

NOTE: not always necessary to use the fully symmetrised proton wave-function, e.g. the first 3 terms are sufficient for calculating the proton magnetic moment

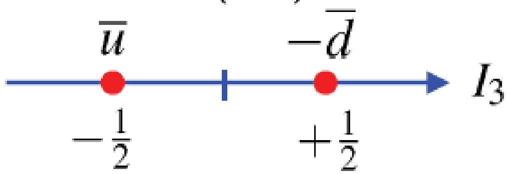
Anti-quarks and Mesons (u and d)

★ The u, d quarks and ū, d̄ anti-quarks are represented as isospin doublets

$$q = \begin{pmatrix} u \\ d \end{pmatrix}$$



$$\bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$$



$$\bar{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \bar{d} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- **Subtle point:** The ordering and the minus sign in the anti-quark doublet ensures that anti-quarks and quarks transform in the same way (see Appendix I). This is necessary if we want physical predictions to be invariant under $u \leftrightarrow d;\ \bar{u} \leftrightarrow \bar{d}$

- Consider the effect of ladder operators on the anti-quark isospin states

e.g

$$T_+\bar{u} = T_+\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\bar{d}$$

- The effect of the ladder operators on anti-particle isospin states are:

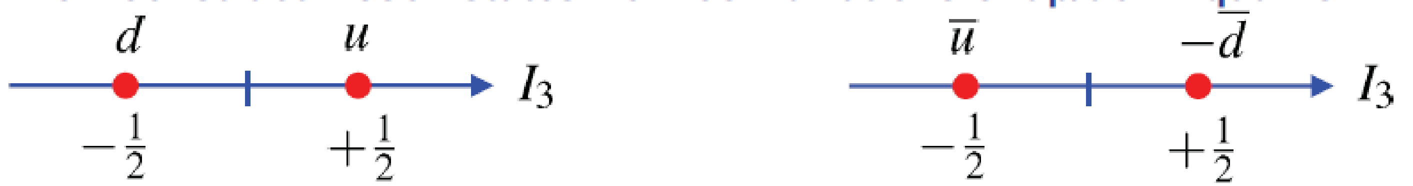
$$T_+\bar{u} = -\bar{d} \qquad T_+\bar{d} = 0 \qquad T_-\bar{u} = 0 \qquad T_-\bar{d} = -\bar{u}$$

Compare with

$$T_+u = 0 \qquad T_+d = u \qquad T_-u = d \qquad T_-d = 0$$

Light ud Mesons

★ Can now construct meson states from combinations of up/down quarks



- Consider the $q\overline{q}$ combinations in terms of isospin

$$|1,+1\rangle = |\tfrac{1}{2},+\tfrac{1}{2}\rangle\overline{|\tfrac{1}{2},+\tfrac{1}{2}\rangle} = -u\overline{d}$$

$$|1,-1\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle\overline{|\tfrac{1}{2},-\tfrac{1}{2}\rangle} = d\overline{u}$$

The bar indicates this is the isospin representation of an anti-quark

To obtain the $I_3 = 0$ states use ladder operators and orthogonality

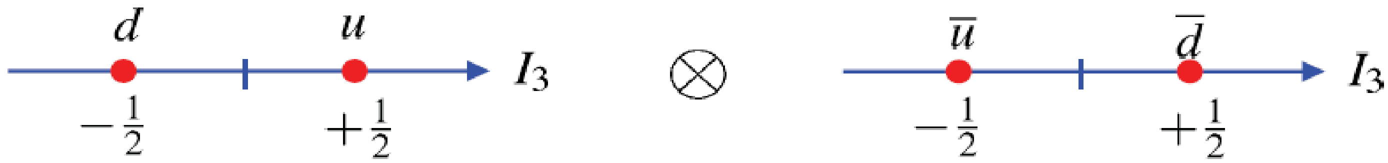
$$\begin{aligned} T_-|1,+1\rangle &= T_-\left[-u\overline{d}\right] \\ \sqrt{2}|1,0\rangle &= -T_-\left[u\right]\overline{d} - uT_-\left[\overline{d}\right] \\ &= -d\overline{d} + u\overline{u} \end{aligned}$$

$$\Rightarrow \quad |1,0\rangle = \frac{1}{\sqrt{2}}\left(u\overline{u} - d\overline{d}\right)$$

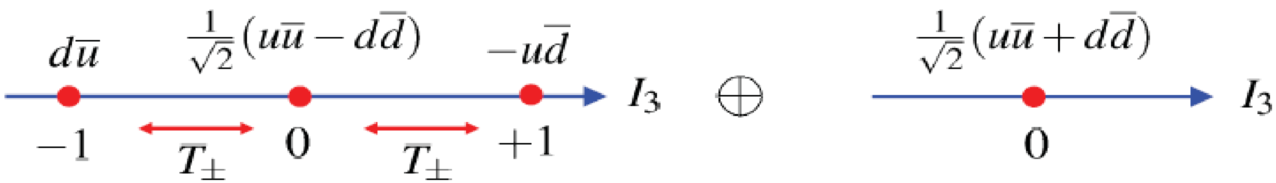
- Orthogonality gives: $|0,0\rangle = \frac{1}{\sqrt{2}}\left(u\overline{u} + d\overline{d}\right)$

Light ud Mesons

★To summarise:



Triplet of $I=1$ states and a singlet $I=0$ state



- You will see this written as $2 \otimes \bar{2} = 3 \oplus 1$

  Quark doublet — Anti-quark doublet

- To show the state obtained from orthogonality with $|1,0\rangle$ is a singlet use ladder operators

$$T_+|0,0\rangle = T_+\frac{1}{\sqrt{2}}(u\bar{u}+d\bar{d}) = \frac{1}{\sqrt{2}}\left(-u\bar{d}+u\bar{d}\right) = 0$$

similarly $\quad T_-|0,0\rangle = 0$

★ A singlet state is a "dead-end" from the point of view of ladder operators

SU(3) flavour

★ Extend these ideas to include the strange quark. Since $m_s > m_u, m_d$ don't have an <u>exact symmetry</u>. But m_s not so very different from m_u, m_d and can treat the strong interaction (and resulting hadron states) as if it were symmetric under $u \leftrightarrow d \leftrightarrow s$

- **NOTE:** any results obtained from this assumption are only approximate as the symmetry is not exact.
- The assumed uds flavour symmetry can be expressed as

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- The 3x3 unitary matrix depends on 9 complex numbers, i.e. 18 real parameters There are 9 constraints from $\hat{U}^\dagger \hat{U} = 1$

⇒ Can form 18 – 9 = 9 linearly independent matrices

These 9 matrices form a U(3) group.

- As before, one matrix is simply the identity multiplied by a complex phase and is of no interest in the context of flavour symmetry
- The remaining 8 matrices have $\det U = 1$ and form an SU(3) group
- The eight matrices (the Hermitian generators) are: $\vec{T} = \frac{1}{2}\vec{\lambda}$ $\hat{U} = e^{i\vec{\alpha}.\vec{T}}$

★ In SU(3) flavour, the three quark states are represented by:

$$u = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad d = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad s = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

★ In SU(3) **uds** flavour symmetry contains SU(2) **ud** flavour symmetry which allows us to write the first three matrices:

$$\lambda_1 = \begin{pmatrix} \sigma_1 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} \sigma_2 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} \sigma_3 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

i.e. **u ↔ d**

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- The third component of isospin is now written $I_3 = \frac{1}{2}\lambda_3$

  with $I_3 u = +\frac{1}{2}u \quad I_3 d = -\frac{1}{2}d \quad I_3 s = 0$

- I_3 "counts the number of up quarks – number of down quarks in a state
- As before, ladder operators $T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$ $\quad d \bullet \longleftarrow T_\pm \longrightarrow \bullet\, u$

- Now consider the matrices corresponding to the **u ↔ s** and **d ↔ s**

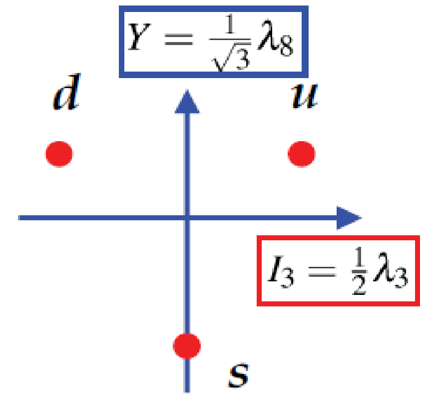
u ↔ s $\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

d ↔ s $\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

- Hence in addition to $\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ have two other traceless diagonal matrices
- However the three diagonal matrices are not be independent.
- Define the eighth matrix, λ_8 , as the linear combination:

$$\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

which specifies the "vertical position" in the 2D plane

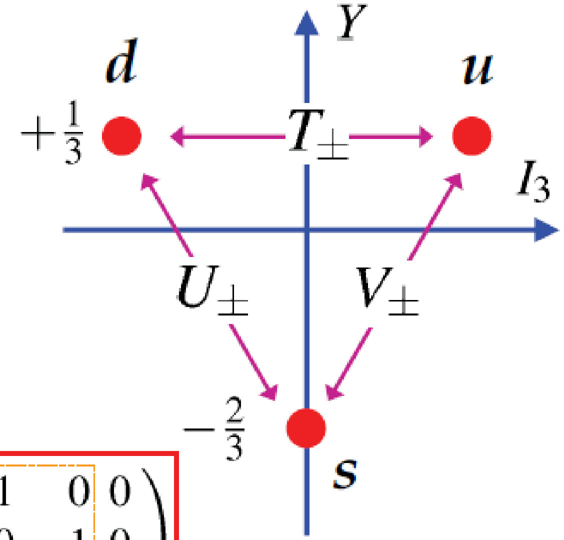


"Only need two axes (quantum numbers) to specify a state in the 2D plane": **(I_3,Y)**

★The other six matrices form six ladder operators which step between the states

$$T_{\pm} = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_{\pm} = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_{\pm} = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$

with $I_3 = \tfrac{1}{2}\lambda_3$ $\quad Y = \tfrac{1}{\sqrt{3}}\lambda_8$



and the eight **Gell-Mann** matrices

u ↔ d

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

u ↔ s

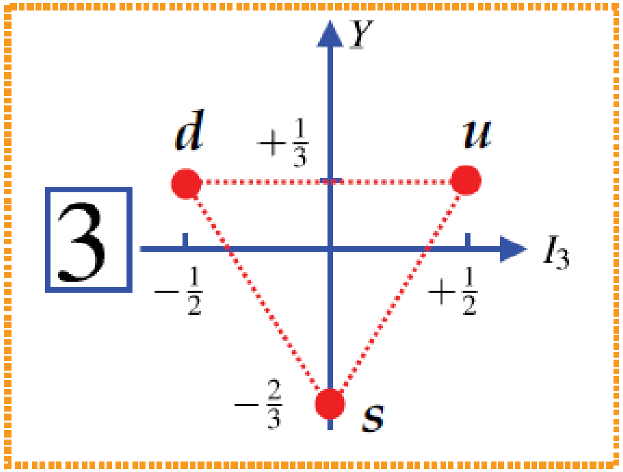
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

d ↔ s

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Quarks and anti-quarks in SU(3) flavour

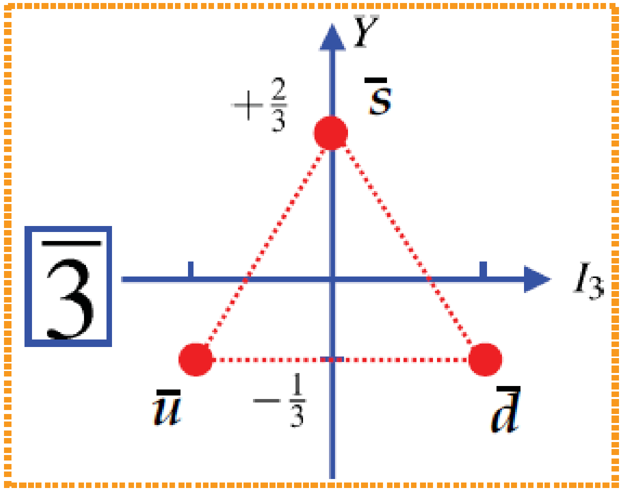


Quarks

$$I_3 u = +\tfrac{1}{2}u; \quad I_3 d = -\tfrac{1}{2}d; \quad I_3 s = 0$$

$$Yu = +\tfrac{1}{3}u; \quad Yd = +\tfrac{1}{3}d; \quad Ys = -\tfrac{2}{3}s$$

- The anti-quarks have opposite **SU(3)** flavour quantum numbers



Anti-Quarks

$$I_3 \bar{u} = -\tfrac{1}{2}\bar{u}; \quad I_3 \bar{d} = +\tfrac{1}{2}\bar{d}; \quad I_3 \bar{s} = 0$$

$$Y\bar{u} = -\tfrac{1}{3}\bar{u}; \quad Y\bar{d} = -\tfrac{1}{3}\bar{d}; \quad Y\bar{s} = +\tfrac{2}{3}\bar{s}$$

SU(3) ladder operators

- SU(3) ***uds*** flavour symmetry contains ***ud***, ***us*** and ***ds*** SU(2) symmetries
- Consider the $u \leftrightarrow s$ symmetry "V-spin" which has the associated $s \rightarrow u$ ladder operator

$$V_+ = \tfrac{1}{2}(\lambda_4 + i\lambda_5) = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \tfrac{i}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with

$$V_+ s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = +u$$

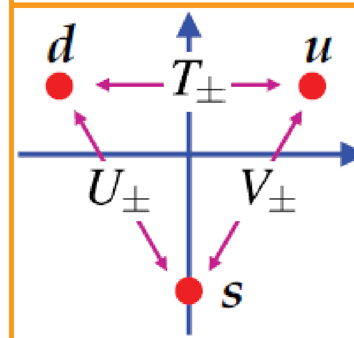
★ The effects of the six ladder operators are:

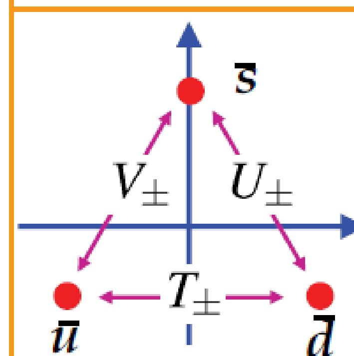
$T_+ d = u; \quad T_- u = d;$	$T_+ \bar{u} = -\bar{d}; \quad T_- \bar{d} = -\bar{u}$
$V_+ s = u; \quad V_- u = s;$	$V_+ \bar{u} = -\bar{s}; \quad V_- \bar{s} = -\bar{u}$
$U_+ s = d; \quad U_- d = s;$	$U_+ \bar{d} = -\bar{s}; \quad U_- \bar{s} = -\bar{d}$

all other combinations give zero

SU(3) LADDER OPERATORS

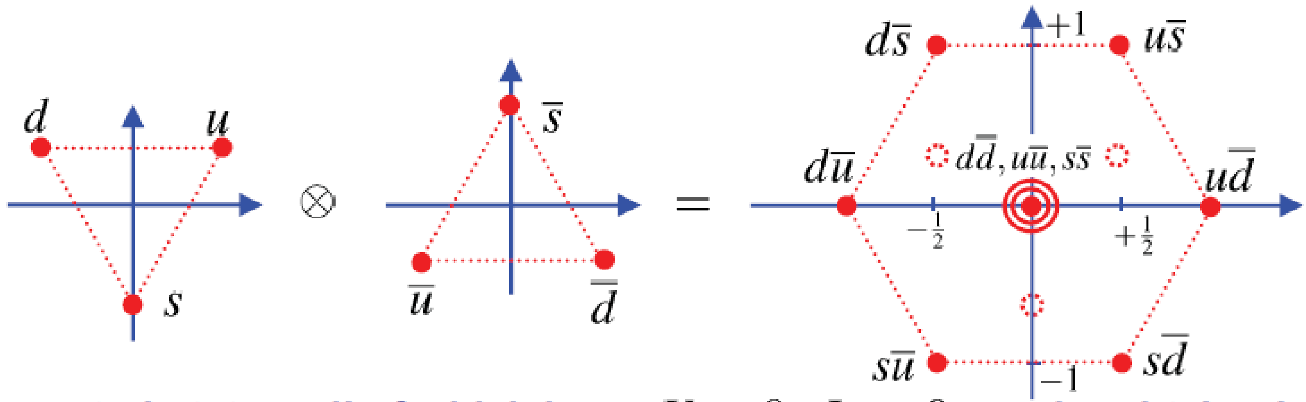
$$T_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$



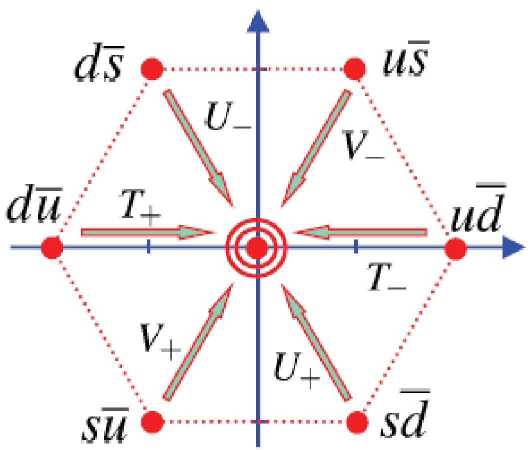


Light (uds) mesons

- Use ladder operators to construct **uds** mesons from the nine possible $q\bar{q}$ states



- The three central states, all of which have $Y = 0;\ I_3 = 0$ can be obtained using the ladder operators and orthogonality. Starting from the outer states can reach the centre in six ways



$$T_+|d\bar{u}\rangle = |u\bar{u}\rangle - |d\bar{d}\rangle \qquad T_-|u\bar{d}\rangle = |d\bar{d}\rangle - |u\bar{u}\rangle$$
$$V_+|s\bar{u}\rangle = |u\bar{u}\rangle - |s\bar{s}\rangle \qquad V_-|u\bar{s}\rangle = |s\bar{s}\rangle - |u\bar{u}\rangle$$
$$U_+|s\bar{d}\rangle = |d\bar{d}\rangle - |s\bar{s}\rangle \qquad U_-|d\bar{s}\rangle = |s\bar{s}\rangle - |d\bar{d}\rangle$$

- Only two of these six states are linearly independent.
- But there are three states with $Y = 0;\ I_3 = 0$
- Therefore one state is not part of the same multiplet, i.e. cannot be reached with ladder ops.

- First form two linearly independent orthogonal states from:

$$|u\bar{u}\rangle - |d\bar{d}\rangle \qquad |u\bar{u}\rangle - |s\bar{s}\rangle \qquad |d\bar{d}\rangle - |s\bar{s}\rangle$$

★ If the SU(3) flavour symmetry were exact, the choice of states wouldn't matter. However, $m_s > m_{u,d}$ and the symmetry is only approximate.

- Experimentally observe three light mesons with m~140 MeV: $\pi^+,\ \pi^0,\ \pi^-$
- Identify one state (the π^0) with the isospin triplet (derived previously)

$$\psi_1 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$

- The second state can be obtained by taking the linear combination of the other two states which is orthogonal to the π^0

$$\psi_2 = \alpha(|u\bar{u}\rangle - |s\bar{s}\rangle) + \beta(|d\bar{d}\rangle - |s\bar{s}\rangle)$$

with orthonormality: $\langle\psi_1|\psi_2\rangle = 0; \quad \langle\psi_2|\psi_2\rangle = 1$

$$\rightarrow \quad \psi_2 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

- The final state (which is not part of the same multiplet) can be obtained by requiring it to be orthogonal to ψ_1 and ψ_2

$$\rightarrow \quad \psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

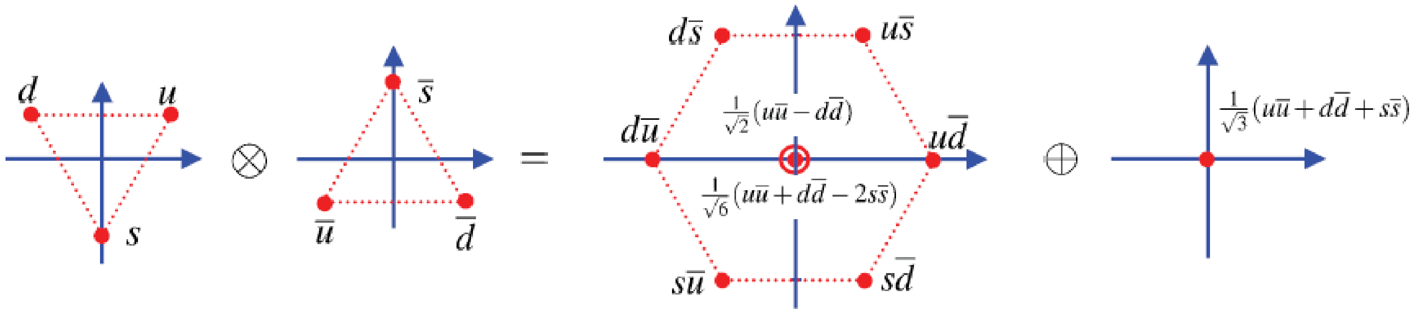
SINGLET

★It is easy to check that ψ_3 is a singlet state using ladder operators

$$T_+\psi_3 = T_-\psi_3 = U_+\psi_3 = U_-\psi_3 = V_+\psi_3 = V_-\psi_3 = 0$$

which confirms that $\psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$ is a "flavourless" singlet

- Therefore the combination of a quark and anti-quark yields nine states which breakdown into an OCTET and a SINGLET



- In the language of group theory: $3 \otimes \bar{3} = 8 \oplus 1$

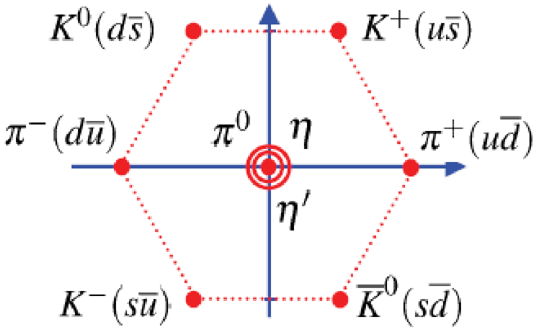
★ Compare with combination of two spin-half particles $2 \otimes 2 = 3 \oplus 1$

TRIPLET of spin-1 states: $|1,-1\rangle,\ |1,0\rangle,\ |1,+1\rangle$

spin-0 SINGLET: $|0,0\rangle$

- These spin triplet states are connected by ladder operators just as the meson ***uds*** octet states are connected by **SU(3)** flavour ladder operators
- The singlet state carries no angular momentum – in this sense the **SU(3)** flavour singlet is "flavourless"

PSEUDOSCALAR MESONS (L=0, S=0, J=0, P= –1)



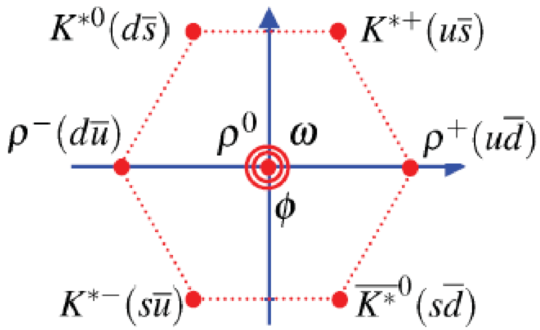
- Because SU(3) flavour is only approximate the physical states with $I_3 = 0,\ Y = 0$ can be mixtures of the octet and singlet states.

Empirically find:

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\eta \approx \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$
$$\eta' \approx \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

← singlet

VECTOR MESONS (L=0, S=1, J=1, P= –1)



- For the vector mesons the physical states are found to be approximately "ideally mixed":

$$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\omega \approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$
$$\phi \approx s\bar{s}$$

MASSES

$\pi^\pm$: 140 MeV	π^0 : 135 MeV
$K^\pm$: 494 MeV	$K^0/\overline{K}^0$: 498 MeV
η : 549 MeV	η' : 958 MeV

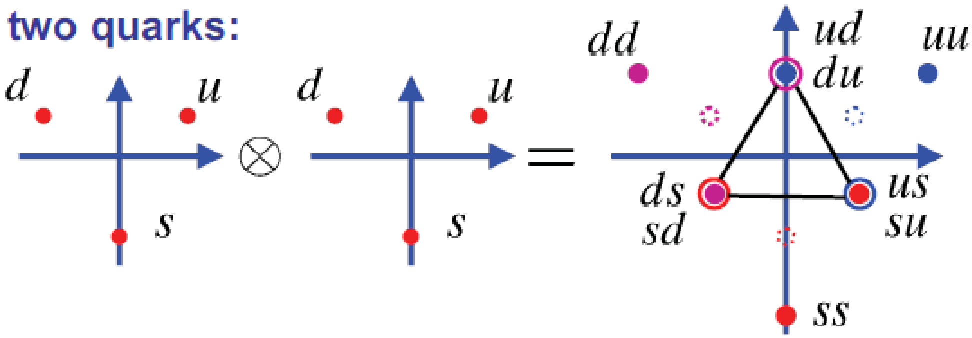
$\rho^\pm$: 770 MeV	ρ^0 : 770 MeV
$K^{*\pm}$: 892 MeV	$K^{*0}/\overline{K^*}^0$: 896 MeV
ω : 782 MeV	ϕ : 1020 MeV

Combining uds Quarks to form Bayrons

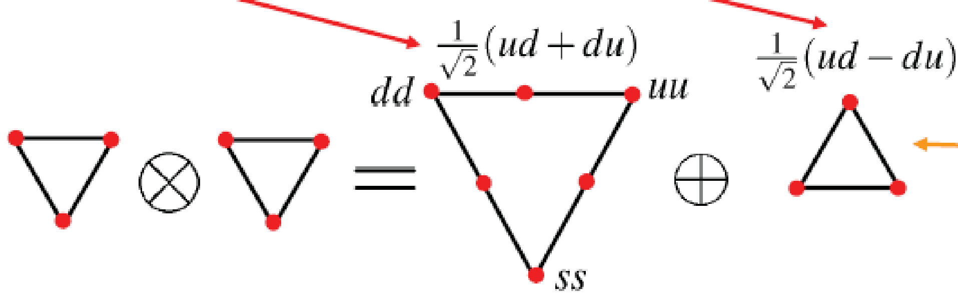
★ Have already seen that constructing Baryon states is a fairly tedious process when we derived the proton wave-function. Concentrate on multiplet structure rather than deriving all the wave-functions.

★ <u>Everything we do here is relevant to the treatment of colour</u>

- First combine two quarks:



★ Yields a symmetric sextet and anti-symmetric triplet: $3 \otimes 3 = 6 \oplus \bar{3}$

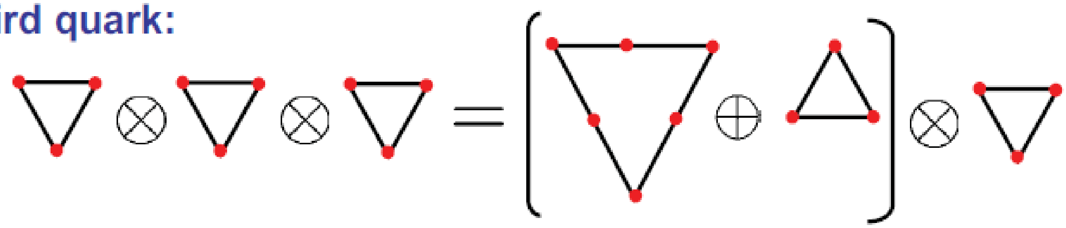


Same "pattern" as the anti-quark representation

SYMMETRIC

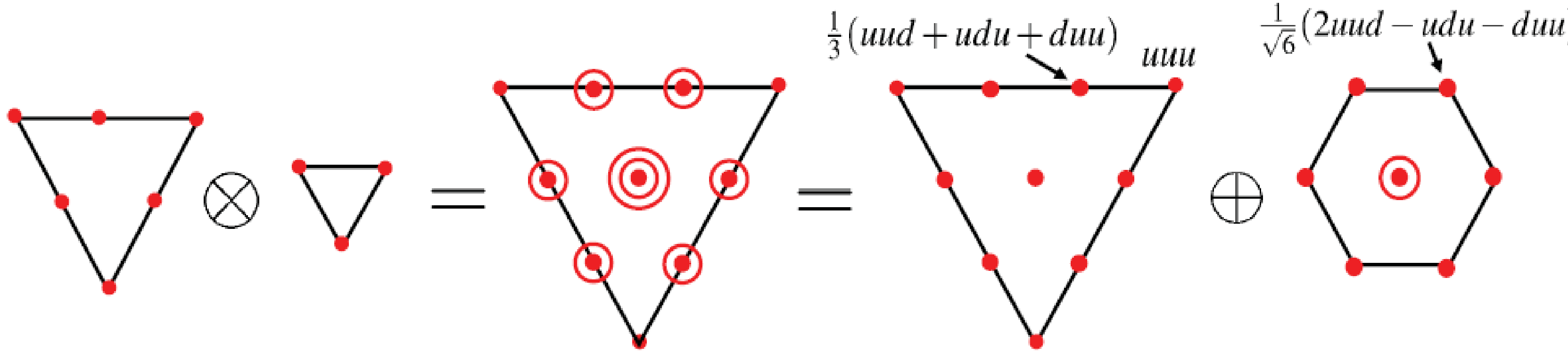
ANTI-SYMMETRIC

- Now add the third quark:



- Best considered in two parts, building on the sextet and triplet. Again concentrate on the multiplet structure (for the wave-functions refer to the discussion of proton wave-function).

❶ Building on the sextet: $3 \otimes 6 = 10 \oplus 8$

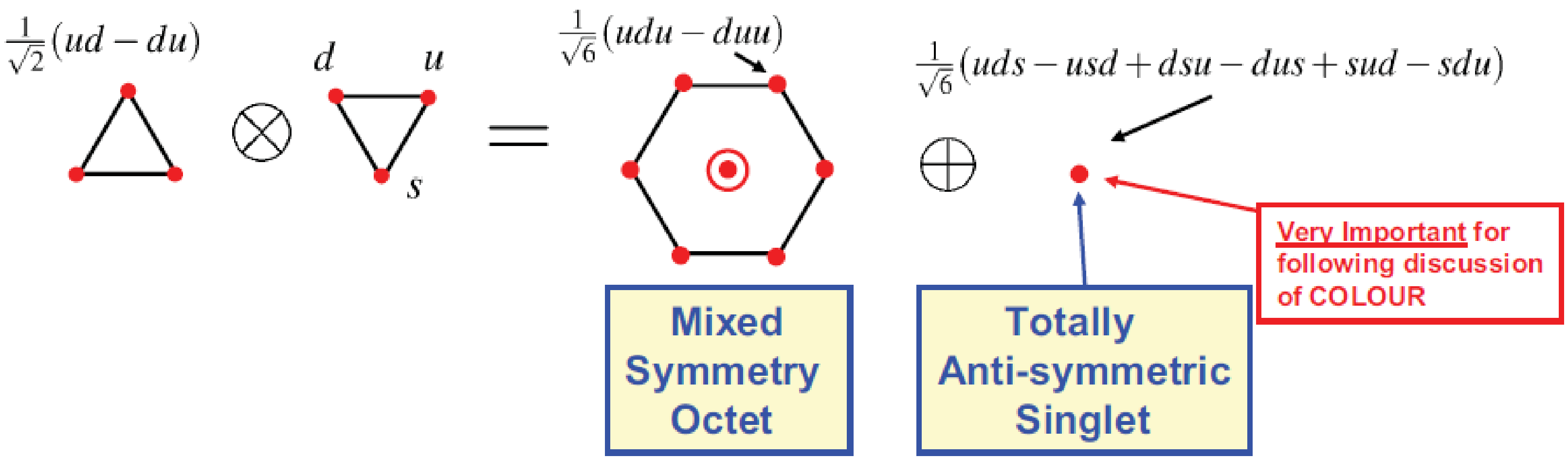


Symmetric Decuplet

Mixed Symmetry Octet

❷ **Building on the triplet:**

- Just as in the case of uds mesons we are combining $\bar{3} \times 3$ and again obtain an octet and a singlet



- Can verify the wave-function $\psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uds - usd + dsu - dus + sud - sdu)$ is a singlet by using ladder operators, e.g.

$$T_+ \psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uus - usu + usu - uus + suu - suu) = 0$$

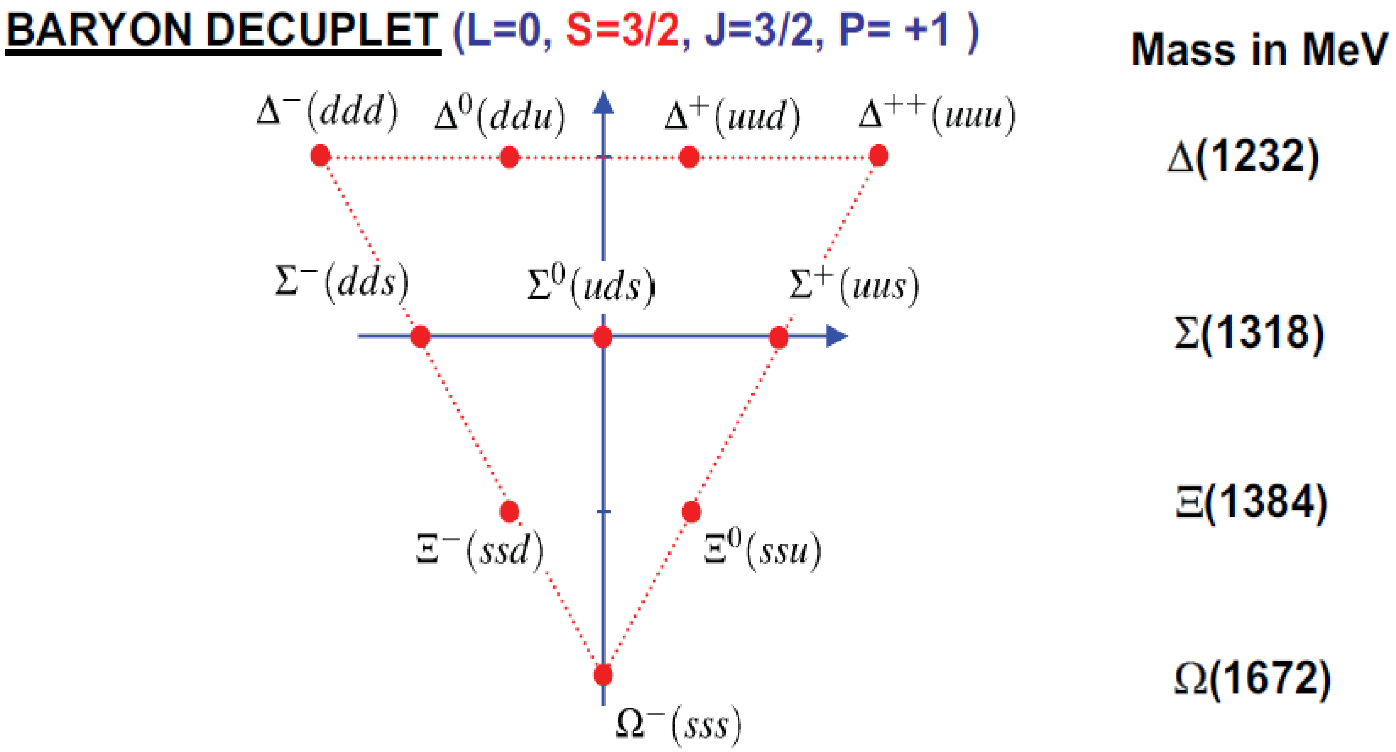
★ In summary, the combination of three ***uds*** quarks decomposes into

$$3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \bar{3}) = 10 \oplus 8 \oplus 8 \oplus 1$$

Baryon decuplet

★ The baryon states (L=0) are:

- the spin 3/2 decuplet of symmetric flavour and symmetric spin wave-functions $\phi(S)\chi(S)$



★ If SU(3) flavour were an exact symmetry all masses would be the same (broken symmetry)

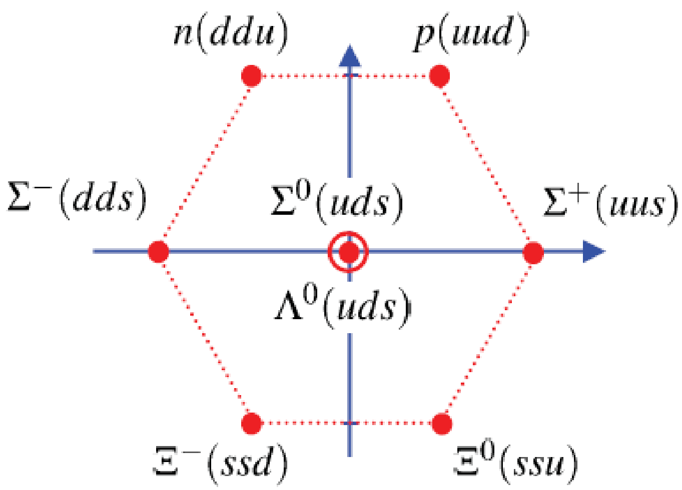
Baryon octet

★ The **spin 1/2 octet** is formed from mixed symmetry flavour and mixed symmetry spin wave-functions

$$\alpha\phi(M_S)\chi(M_S)+\beta\phi(M_A)\chi(M_A)$$

See previous discussion proton for how to obtain wave-functions

<u>BARYON OCTET</u> (L=0, S=1/2, J=1/2, P= +1)



Mass in MeV

939

Σ(1193)
Λ(1116)

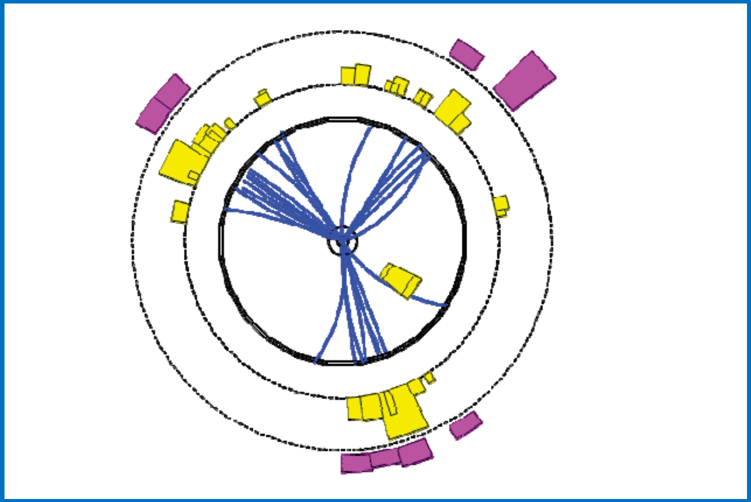
Ξ(1318)

★ **NOTE:** Cannot form a totally symmetric wave-function based on the anti-symmetric flavour singlet as there no totally anti-symmetric spin wave-function for 3 quarks

Summary

- ★ Considered SU(2) **ud** and SU(3) **uds** flavour symmetries
- ★ Although these flavour symmetries are only approximate can still be used to explain observed multiplet structure for mesons/baryons
- ★ In case of SU(3) flavour symmetry results, e.g. predicted wave-functions should be treated with a pinch of salt as $m_s \neq m_{u/d}$
- ★ Introduced idea of singlet states being "spinless" or "flavourless"

Quantum Chromodynamics



The local gauge principle

★ All the interactions between fermions and spin-1 bosons in the SM are specified by the principle of LOCAL GAUGE INVARIANCE

★ To arrive at QED, require physics to be invariant under the local phase transformation of particle wave-functions

$$\psi \rightarrow \psi' = \psi e^{iq\chi(x)}$$

★ Note that the change of phase depends on the space-time coordinate: $\chi(t, \vec{x})$

- Under this transformation the Dirac Equation transforms as

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \quad \Rightarrow \quad i\gamma^\mu(\partial_\mu + iq\partial_\mu \chi)\psi - m\psi = 0$$

- To make "physics", i.e. the Dirac equation, invariant under this local phase transformation FORCED to introduce a massless gauge boson, A_μ .

\+ The Dirac equation has to be modified to include this new field:

$$i\gamma^\mu(\partial_\mu - qA_\mu)\psi - m\psi = 0$$

- The modified Dirac equation is invariant under local phase transformations if:

$$A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \chi$$

Gauge Invariance

★ For physics to remain unchanged – must have GAUGE INVARIANCE of the new field, i.e. physical predictions unchanged for $A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi$

★Hence the principle of invariance under local phase transformations completely specifies the interaction between a fermion and the gauge boson (i.e. photon):

$$i\gamma^\mu(\partial_\mu \psi - qA_\mu)\psi - m\psi = 0$$

⇨ interaction vertex: $i\gamma^\mu qA_\mu$

⇨ QED !

★ The local phase transformation of QED is a unitary U(1) transformation

$$\psi \to \psi' = \hat{U}\psi \quad \text{i.e.} \quad \psi \to \psi' = \psi e^{iq\chi(x)} \quad \text{with} \quad U^\dagger U = 1$$

Now extend this idea…

From QED to QCD

★ Suppose there is another fundamental symmetry of the universe, say "invariance under SU(3) local phase transformations"

- i.e. require invariance under $\psi \to \psi' = \psi e^{ig\vec{\lambda}.\vec{\theta}(x)}$ where

$\vec{\lambda}$ are the eight 3x3 Gell-Mann matrices

$\vec{\theta}(x)$ are 8 functions taking different values at each point in space-time

⇨ 8 spin-1 gauge bosons

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

wave function is now a vector in COLOUR SPACE

⇨ **QCD !**

★ QCD is fully specified by require invariance under SU(3) local phase transformations

> Corresponds to rotating states in colour space about an axis whose direction is different at every space-time point

⇨ interaction vertex: $-\frac{1}{2} i g_s \lambda^a \gamma^\mu$

★ Predicts 8 massless gauge bosons – the gluons (one for each λ)

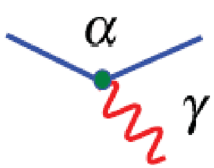
★ Also predicts exact form for interactions between gluons, i.e. the 3 and 4 gluon vertices – the details are beyond the level of this course

Colour in QCD

★ The theory of the strong interaction, Quantum Chromodynamics (QCD), is very similar to QED but with 3 conserved "colour" charges

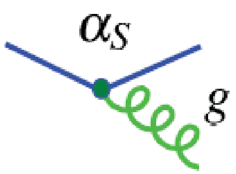
In QED:

- the electron carries one unit of charge $-e$
- the anti-electron carries one unit of anti-charge $+e$
- the force is mediated by a massless "gauge boson" – the photon



In QCD:

- quarks carry colour charge: r, g, b
- anti-quarks carry anti-charge: $\bar{r}, \bar{g}, \bar{b}$
- The force is mediated by massless gluons



★ In QCD, the strong interaction is invariant under rotations in colour space

$$r \leftrightarrow b;\ r \leftrightarrow g;\ b \leftrightarrow g$$

i.e. the same for all three colours

⇒ **SU(3) colour symmetry**

- This is an exact symmetry, unlike the approximate uds flavour symmetry

★ Represent r, g, b SU(3) colour states by:

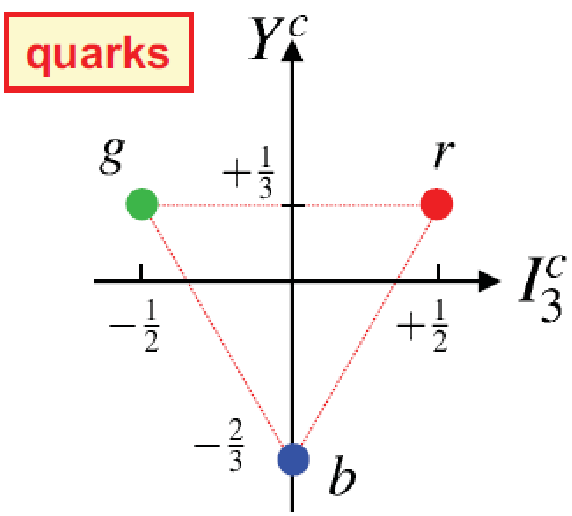
$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

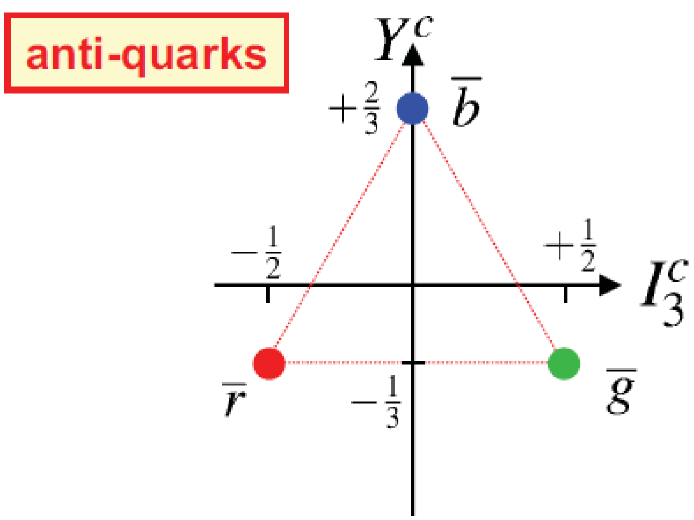
★ Colour states can be labelled by two quantum numbers:

- I_3^c colour isospin
- Y^c colour hypercharge

Exactly analogous to labelling u,d,s flavour states by I_3 and Y

★ Each quark (anti-quark) can have the following colour quantum numbers:





Colour Confinement

★ It is believed (although not yet proven) that all observed free particles are "colourless"

- i.e. never observe a free quark (which would carry colour charge)
- consequently quarks are always found in bound states colourless hadrons

★ **Colour Confinement Hypothesis:**

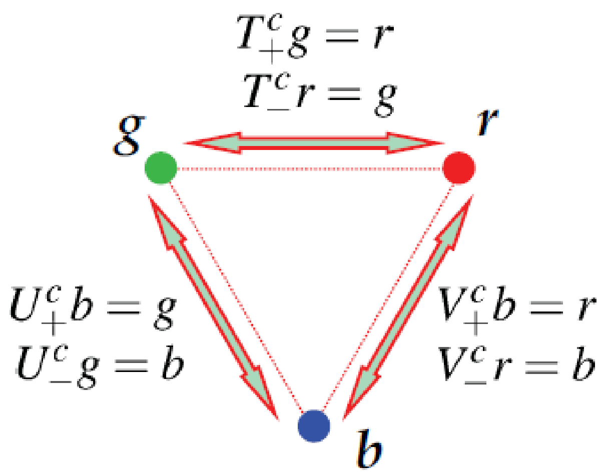
only colour singlet states can exist as free particles

★ All hadrons must be "colourless" i.e. colour singlets

★ To construct colour wave-functions for hadrons can apply results for SU(3) flavour symmetry to SU(3) colour with replacement

$$u \to r$$
$$d \to g$$
$$s \to b$$

★ just as for uds flavour symmetry can define colour ladder operators



Colour Singlets

★ It is important to understand what is meant by a singlet state

★ Consider spin states obtained from two spin 1/2 particles.

- Four spin combinations: $\uparrow\uparrow,\ \uparrow\downarrow,\ \downarrow\uparrow,\ \downarrow\downarrow$
- Gives four eigenstates of $\hat{S}^2,\ \hat{S}_z$ $\quad (2\otimes 2 = 3\oplus 1)$

$$|1,+1\rangle = \uparrow\uparrow$$
$$|1,0\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$$
$$|1,-1\rangle = \downarrow\downarrow$$

spin-1 triplet

$$\oplus\ |0,0\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

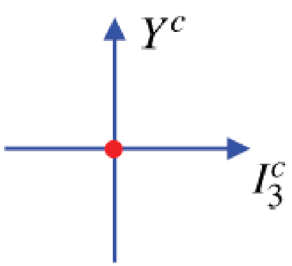
spin-0 singlet

★ The singlet state is "spinless": it has zero angular momentum, is invariant under SU(2) spin transformations and spin ladder operators yield zero

$$S_{\pm}|0,0\rangle = 0$$

★ In the same way COLOUR SINGLETS are "colourless" combinations:

- they have zero colour quantum numbers $I_3^c = 0,\ Y^c = 0$
- invariant under SU(3) colour transformations
- ladder operators $T_{\pm},\ U_{\pm},\ V_{\pm}$ all yield zero

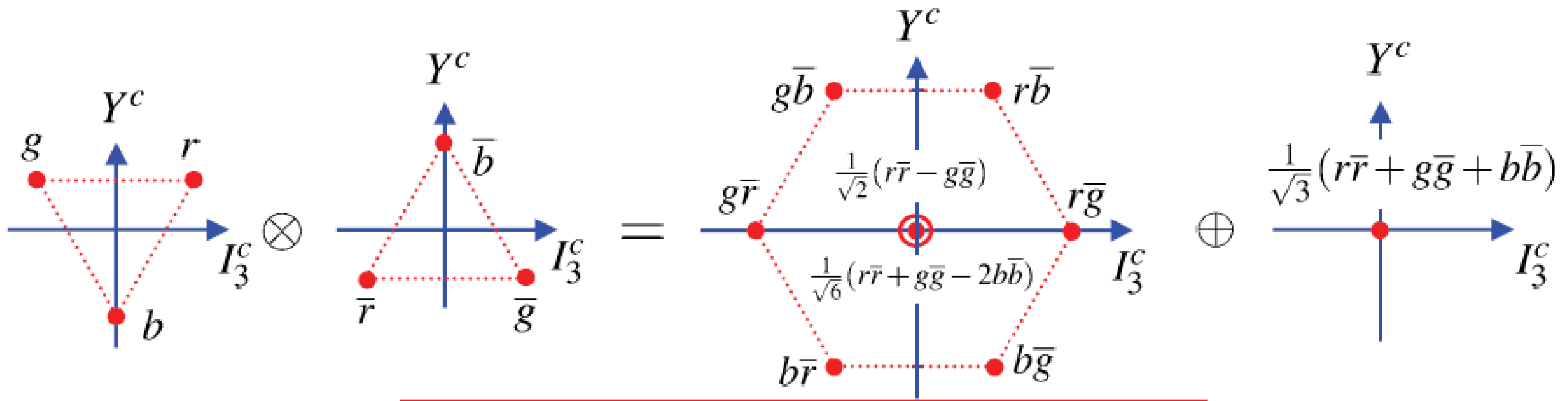


★ NOT sufficient to have $I_3^c = 0,\ Y^c = 0$: does not mean that state is a singlet

Meson Colour Wave-function

★ Consider colour wave-functions for $q\bar{q}$

★ The combination of colour with anti-colour is mathematically identical to construction of meson wave-functions with uds flavour symmetry



➡ **Coloured octet and a colourless singlet**

- Colour confinement implies that hadrons only exist in colour singlet states so the colour wave-function for mesons is:

$$\psi_c^{q\bar{q}} = \frac{1}{\sqrt{3}}(r\bar{r}+g\bar{g}+b\bar{b})$$

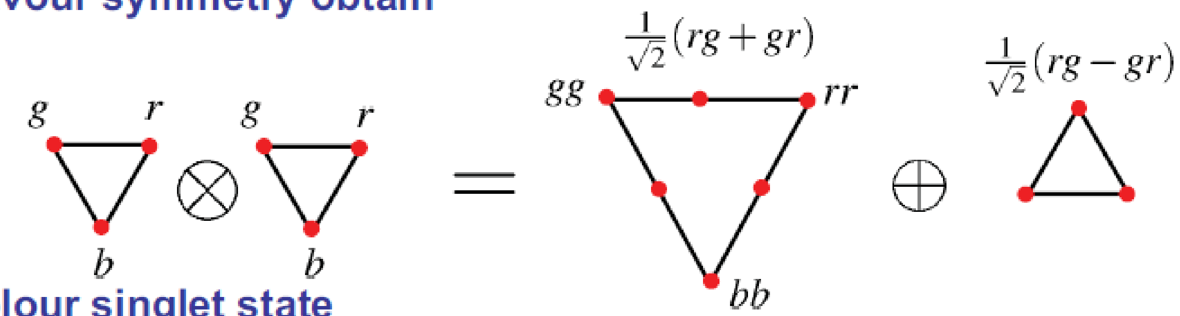
★ Can we have a $qq\bar{q}$ state ? i.e. by adding a quark to the above octet can we form a state with $Y^c = 0;\ I_3^c = 0$. The answer is clear no.

➡ $qq\bar{q}$ bound states do not exist in nature.

Baryon Colour Wave-function

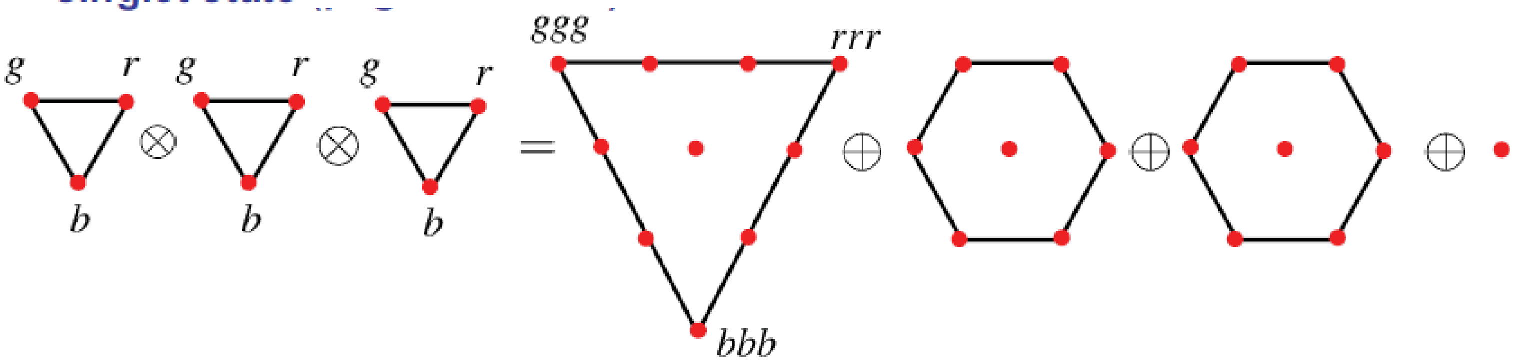
★ Do **qq** bound states exist ? This is equivalent to asking whether it possible to form a colour singlet from two colour triplets ?

- Following the discussion of construction of baryon wave-functions in SU(3) flavour symmetry obtain



- No **qq** colour singlet state
- Colour confinement ➡ bound states of **qq** do not exist

★ BUT combination of three quarks (three colour triplets) gives a colour singlet state



★ The singlet colour wave-function is:

$$\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr)$$

Check this is a colour singlet…

- It has $I_3^c = 0,\ Y^c = 0$: a necessary but not sufficient condition
- Apply ladder operators, e.g. T_+ (recall $T_+ g = r$)

$$T_+\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rrb - rbr + rbr - rrb + brr - brr) = 0$$

- Similarly $T_-\psi_c^{qqq} = 0;\quad V_\pm\psi_c^{qqq} = 0;\quad U_\pm\psi_c^{qqq} = 0;$

★ Colourless singlet - therefore **qqq** bound states exist !

⇨ **Anti-symmetric colour wave-function**

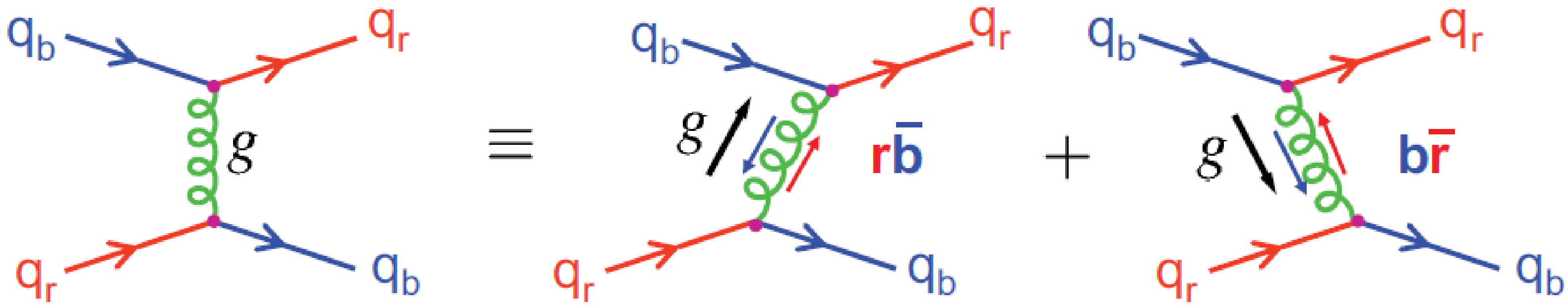
<u>Allowed Hadrons</u> i.e. the possible colour singlet states

- $q\bar{q},\ qqq$ — **Mesons and Baryons**
- $q\bar{q}q\bar{q},\ qqqq\bar{q}$ — **Exotic states, e.g. pentaquarks**

To date all confirmed hadrons are either **mesons** or **baryons**. However, some recent (but not entirely convincing) "evidence" for pentaquark states

Gluons

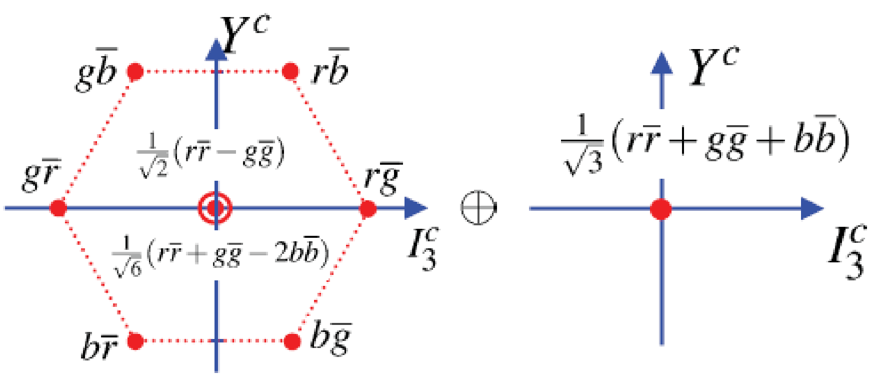
★ In QCD quarks interact by exchanging virtual massless gluons, e.g.



★ Gluons carry colour and anti-colour, e.g.

★ Gluon colour wave-functions (colour + anti-colour) are the same as those obtained for mesons (also colour + anti-colour)

⇨ OCTET + "COLOURLESS" SINGLET



★ So we might expect 9 physical gluons:

OCTET: $r\bar{g},\ r\bar{b},\ g\bar{r},\ g\bar{b},\ b\bar{r},\ b\bar{g},\ \frac{1}{\sqrt{2}}(r\bar{r}-g\bar{g}),\ \frac{1}{\sqrt{6}}(r\bar{r}+g\bar{g}-2b\bar{b})$

SINGLET: $\frac{1}{\sqrt{3}}(r\bar{r}+g\bar{g}+b\bar{b})$

★ **BUT**, colour confinement hypothesis:

only colour singlet states can exist as free particles	⇨	**Colour singlet gluon** would be unconfined. It would behave like a strongly interacting photon ➡ infinite range Strong force.

★ **Empirically**, the strong force is short range and therefore know that the physical gluons are confined. The colour singlet state does not exist in nature !

NOTE: this is not entirely ad hoc. In the context of gauge field theory (see minor option) the strong interaction arises from a fundamental SU(3) symmetry. The gluons arise from the generators of the symmetry group (the Gell-Mann λ matrices). There are 8 such matrices ➡ 8 gluons.
Had nature "chosen" a U(3) symmetry, would have 9 gluons, the additional gluon would be the colour singlet state and QCD would be an unconfined long-range force.

NOTE: the "gauge symmetry" determines the exact nature of the interaction

➡ FEYNMAN RULES

Gluon-Gluon interactions

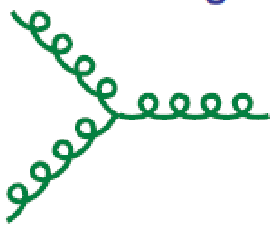
★ In QED the photon does not carry the charge of the EM interaction (photons are electrically neutral)

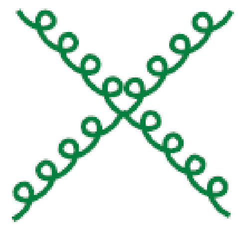
★ In contrast, in QCD the gluons do carry colour charge

⇒ **Gluon Self-Interactions**

★ Two new vertices (no QED analogues)

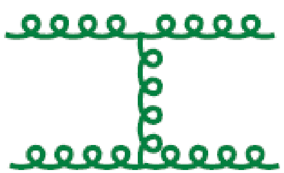
triple-gluon vertex

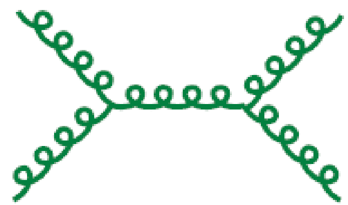




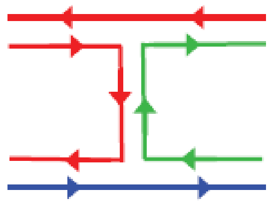
quartic-gluon vertex

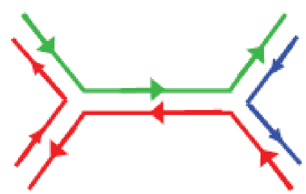
★ In addition to quark-quark scattering, therefore can have gluon-gluon scattering





e.g. possible way of arranging the colour flow



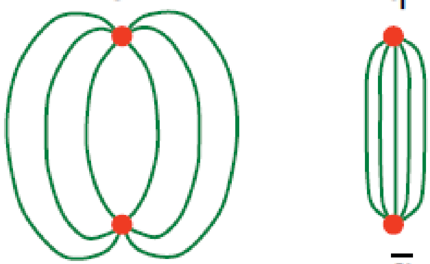


Gluon self-interactions and Confinement

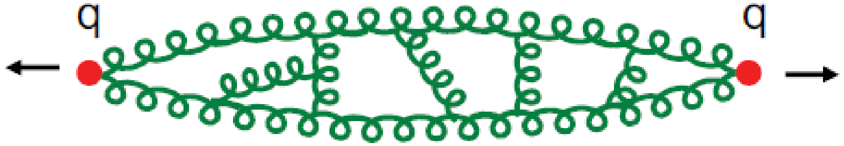
★ Gluon self-interactions are believed to give rise to colour confinement

★ Qualitative picture:

- Compare QED with QCD
- In QCD "gluon self-interactions squeeze lines of force into a flux tube"



★ What happens when try to separate two coloured objects e.g. $q\bar{q}$



- Form a flux tube of interacting gluons of approximately constant energy density $\sim 1\,\mathrm{GeV/fm}$

$$\Rightarrow \quad V(r) \sim \lambda r$$

- Require infinite energy to separate coloured objects to infinity
- Coloured quarks and gluons are always confined within colourless states
- In this way QCD provides a plausible explanation of confinement – but not yet proven (although there has been recent progress with Lattice QCD)

Hadronisation and jets

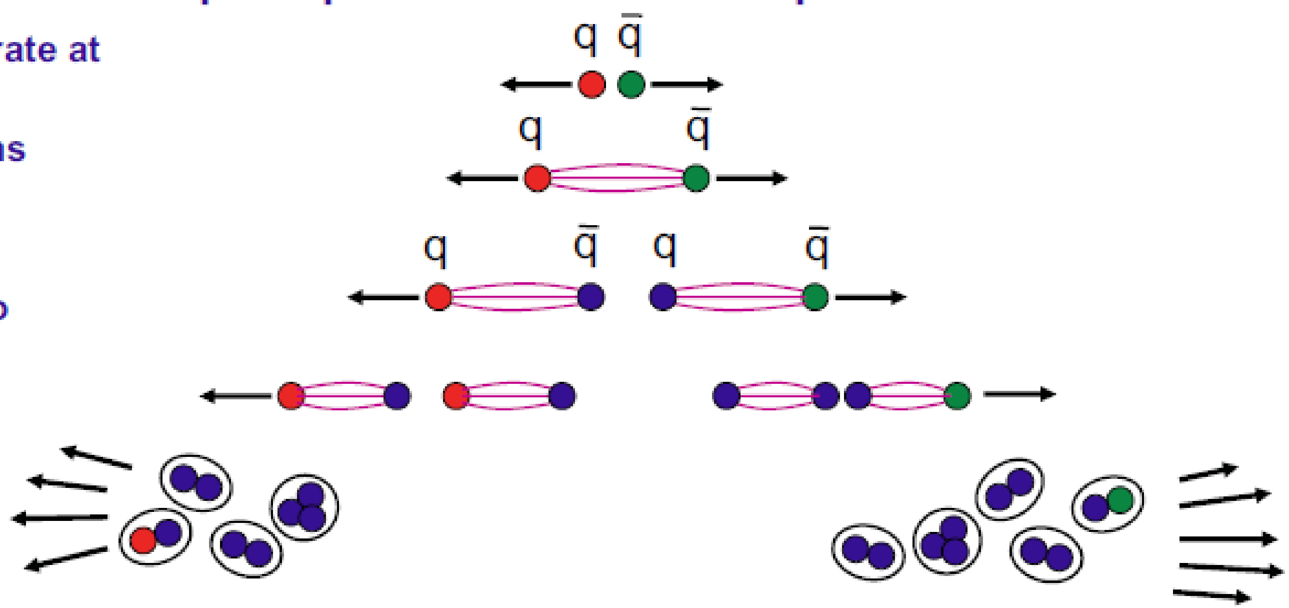
★ Consider a quark and anti-quark produced in electron positron annihilation

i) Initially Quarks separate at high velocity

ii) Colour flux tube forms between quarks

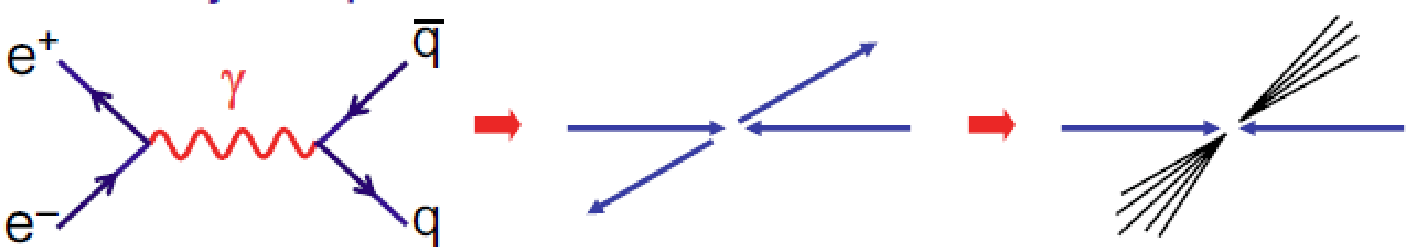
iii) Energy stored in the flux tube sufficient to produce $q\bar{q}$ pairs

iv) Process continues until quarks pair up into jets of colourless hadrons



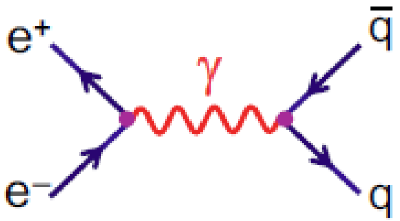
★ This process is called hadronisation. It is not (yet) calculable.

★ The main consequence is that at collider experiments quarks and gluons observed as jets of particles



QCD and Colour in e+e- collisions

★ e^+e^- colliders are an excellent place to study QCD

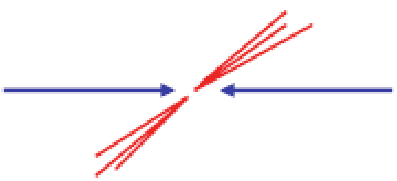


★ Well defined production of quarks
- QED process well-understood
- no need to know parton structure functions
- \+ experimentally very clean – no proton remnants

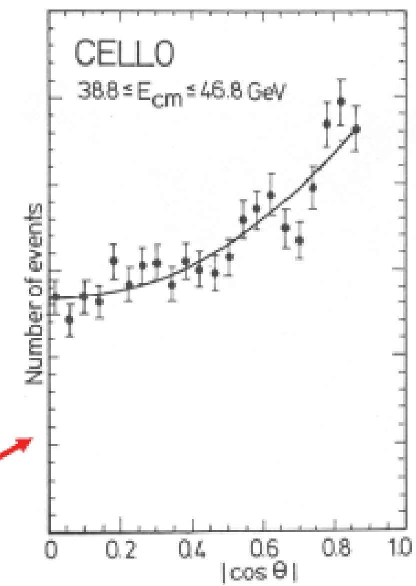
★ expressions for the $e^+e^- \rightarrow \mu^+\mu^-$ cross-section

$$\sigma = \frac{4\pi\alpha^2}{3s} \qquad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

- In e^+e^- collisions produce all quark flavours for which $\sqrt{s} > 2m_q$
- In general, i.e. unless producing a $q\bar{q}$ bound state, produce jets of hadrons
- Usually can't tell which jet came from the quark and came from anti-quark



★ Angular distribution of jets $\propto (1+\cos^2\theta)$



H.J.Behrend et al., Phys Lett 183B (1987) 400

➡ **Quarks are spin ½**

★ Colour is conserved and quarks are produced as $r\bar{r},\ g\bar{g},\ b\bar{b}$

★ For a single quark flavour and single colour

$$\sigma(e^+e^- \to q_i\bar{q}_i) = \frac{4\pi\alpha^2}{3s}Q_q^2$$

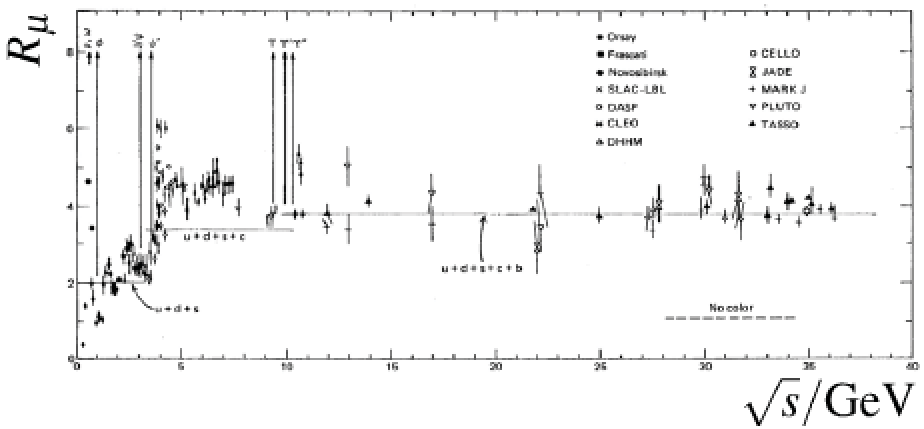
- Experimentally observe jets of hadrons:

$$\sigma(e^+e^- \to \text{hadrons}) = 3\sum_{u,d,s,..}\frac{4\pi\alpha^2}{3s}Q_q^2$$

Factor 3 comes from colours

- Usual to express as ratio compared to $\sigma(e^+e^- \to \mu^+\mu^-)$

$$R_\mu = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_{u,d,s,..}Q_q^2$$



u,d,s: $R_\mu = 3\times(\frac{1}{9}+\frac{4}{9}+\frac{1}{9}) = 2$

u,d,s,c: $R_\mu = \frac{10}{3}$

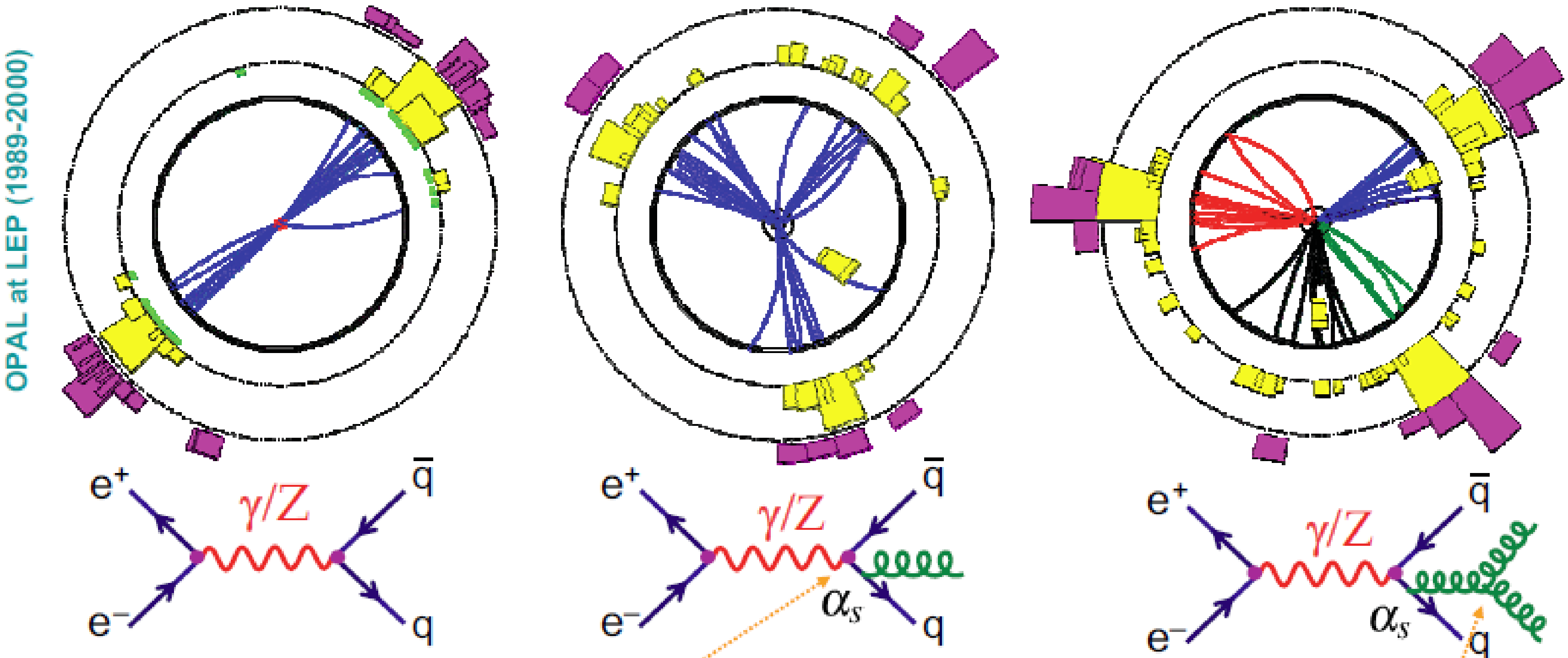
u,d,s,c,b: $R_\mu = \frac{11}{3}$

★ Data consistent with expectation with factor 3 from colour

Jets production in e+e- collisions

★ e^+e^- colliders are also a good place to study gluons

$e^+e^- \rightarrow q\bar{q} \rightarrow$ 2jets $\quad$ $e^+e^- \rightarrow q\bar{q}g \rightarrow$ 3jets $\quad$ $e^+e^- \rightarrow q\bar{q}gg \rightarrow$ 4jets



Experimentally:

- Three jet rate ➡ measurement of α_s
- Angular distributions ➡ gluons are spin-1
- Four-jet rate and distributions ➡ QCD has an underlying SU(3) symmetry

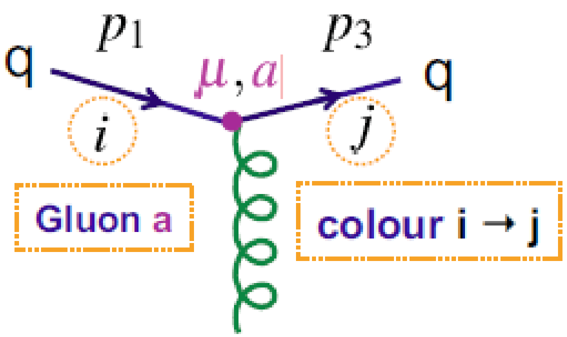
Quark-gluon interaction

- Representing the colour part of the fermion wave-functions by:

$$r = c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad g = c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad b = c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Particle wave-functions $u(p) \longrightarrow c_i u(p)$
- The QCD qqg vertex is written:

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$



- Only difference w.r.t. QED is the insertion of the 3x3 SU(3) Gell-Mann matrices
- Isolating the colour part:

$$c_j^\dagger \lambda^a c_i = c_j^\dagger \begin{pmatrix} \lambda^a_{1i} \\ \lambda^a_{2i} \\ \lambda^a_{3i} \end{pmatrix} = \lambda^a_{ji}$$

- Hence the fundamental quark - gluon QCD interaction can be written

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Feynman rules for QCD

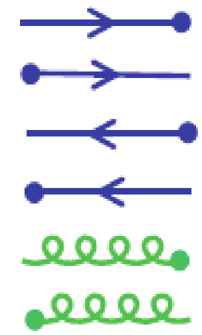
- **External Lines**

spin 1/2:

incoming quark	$u(p)$	
outgoing quark	$\overline{u}(p)$	
incoming anti-quark	$\overline{v}(p)$	
outgoing anti-quark	$v(p)$	

spin 1:

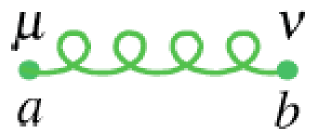
incoming gluon	$\varepsilon^{\mu}(p)$	
outgoing gluon	$\varepsilon^{\mu}(p)^{*}$	



- **Internal Lines (propagators)**

spin 1 gluon

$$\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}$$

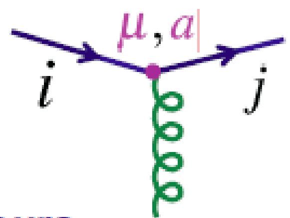


a, b = 1,2,…,8 are gluon colour indices

- **Vertex Factors**

spin 1/2 quark

$$-ig_s\tfrac{1}{2}\lambda^{a}_{ji}\gamma^{\mu}$$



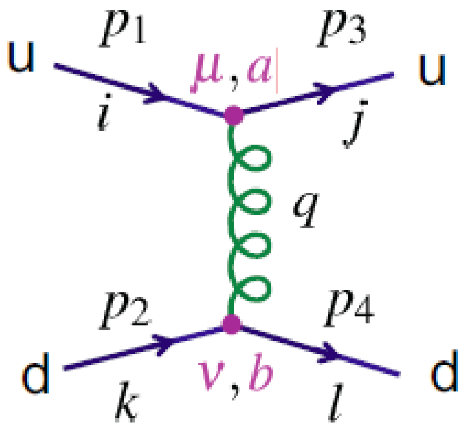
i, j = 1,2,3 are quark colours,

λ^{a} a = 1,2,..8 are the Gell-Mann SU(3) matrices

- **+ 3 gluon and 4 gluon interaction vertices**
- **Matrix Element** $-iM$ = product of all factors

Matrix element for quark-quark scattering

★ Consider QCD scattering of an up and a down quark



- The incoming and out-going quark colours are labelled by $i, j, k, l = \{1,2,3\}$ (or $\{r,g,b\}$)
- In terms of colour this scattering is
$$ik \to jl$$
- The 8 different gluons are accounted for by the colour indices $a, b = 1, 2, ..., 8$
- NOTE: the δ-function in the propagator ensures $a = b$, i.e. the gluon "emitted" at a is the same as that "absorbed" at b

★ Applying the Feynman rules:

$$-iM = [\overline{u}_u(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u_u(p_1)]\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}\left[\overline{u}_d(p_4)\{-\tfrac{1}{2}ig_s\lambda^b_{lk}\gamma^\nu\}u_d(p_2)\right]$$

where summation over a and b (and μ and ν) is implied.

★ Summing over a and b using the δ-function gives:

$$M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]$$

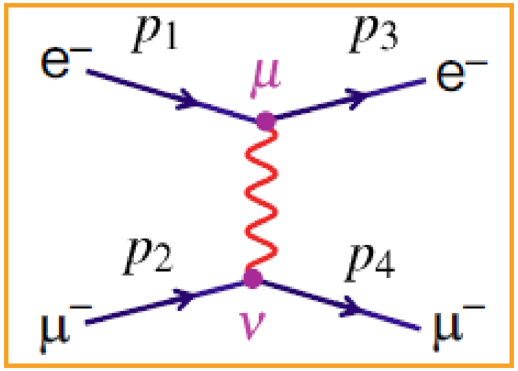
Sum over all 8 gluons (repeated indices)

QCD vs QED

QED

$$-iM = [\overline{u}(p_3)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_4)ie\gamma^\nu u(p_2)]$$

$$M = -e^2\frac{1}{q^2}g_{\mu\nu}[\overline{u}(p_3)\gamma^\mu u(p_1)][\overline{u}(p_4)\gamma^\nu u(p_2)]$$

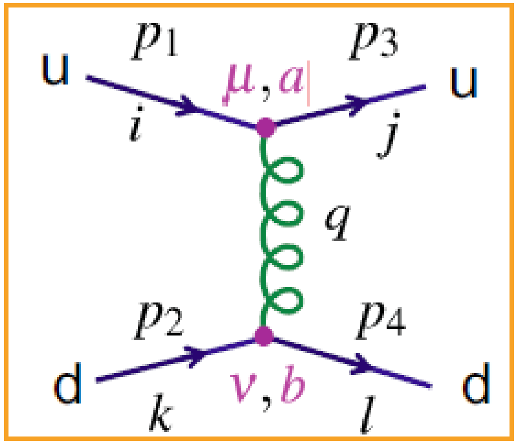


QCD

$$M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]$$

★ **QCD Matrix Element = QED Matrix Element with:**

- $e^2 \to g_s^2$ **or equivalently** $\alpha = \frac{e^2}{4\pi} \to \alpha_s = \frac{g_s^2}{4\pi}$



+ **QCD Matrix Element includes an additional "colour factor"**

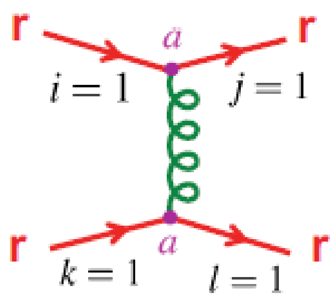
$$C(ik \to jl) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{lk}$$

Evaluation of QCD colour factors

- QCD colour factors reflect the gluon states that are involved

$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
Gluons: $r\bar{g}, g\bar{r}$	$r\bar{b}, b\bar{r}$	$g\bar{b}, b\bar{g}$	$\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$

❶ Configurations involving a single colour



- Only matrices with non-zero entries in 11 position are involved

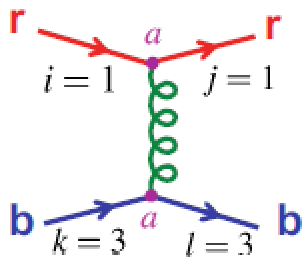
$$C(rr \to rr) = \frac{1}{4}\sum_{a=1}^{8} \lambda_{11}^{a}\lambda_{11}^{a} = \frac{1}{4}(\lambda_{11}^{3}\lambda_{11}^{3} + \lambda_{11}^{8}\lambda_{11}^{8})$$

$$= \frac{1}{4}\left(1 + \frac{1}{3}\right) = \frac{1}{3}$$

Similarly find

$$\boxed{C(rr \to rr) = C(gg \to gg) = C(bb \to bb) = \frac{1}{3}}$$

❷ Other configurations where quarks don't change colour   e.g. $rb \rightarrow rb$



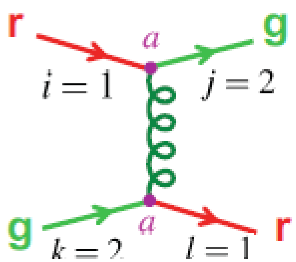
- Only matrices with non-zero entries in 11 and 33 position are involved

$$C(rb \rightarrow rb) = \frac{1}{4}\sum_{a=1}^{8}\lambda_{11}^{a}\lambda_{33}^{a} = \frac{1}{4}(\lambda_{11}^{8}\lambda_{33}^{8})$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{3}}\cdot\frac{-2}{\sqrt{3}}\right) = -\frac{1}{6}$$

Similarly $C(rb \rightarrow rb) = C(rg \rightarrow rg) = C(gr \rightarrow gr) = C(gb \rightarrow gb) = C(br \rightarrow br) = C(bg \rightarrow bg) = -\frac{1}{6}$

❸ Configurations where quarks swap colours   e.g. $rg \rightarrow gr$



- Only matrices with non-zero entries in 12 and 21 position are involved

Gluons $r\bar{g}$, $g\bar{r}$

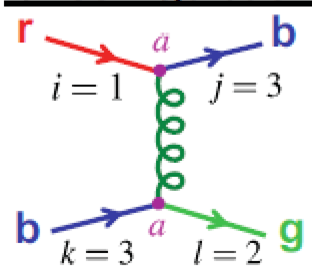
$$C(rg \rightarrow gr) = \frac{1}{4}\sum_{a=1}^{8}\lambda_{21}^{a}\lambda_{12}^{a} = \frac{1}{4}(\lambda_{21}^{1}\lambda_{12}^{1} + \lambda_{21}^{2}\lambda_{12}^{2})$$

$$= \frac{1}{4}(i(-i)+1) = \frac{1}{2}$$

$\hat{T}_{+}^{(ij)}\hat{T}_{-}^{(kl)}$

$$C(rb \rightarrow br) = C(rg \rightarrow gr) = C(gr \rightarrow rg) = C(gb \rightarrow bg) = C(br \rightarrow rb) = C(bg \rightarrow gb) = \frac{1}{2}$$

❹ Configurations involving 3 colours   e.g. $rb \rightarrow bg$



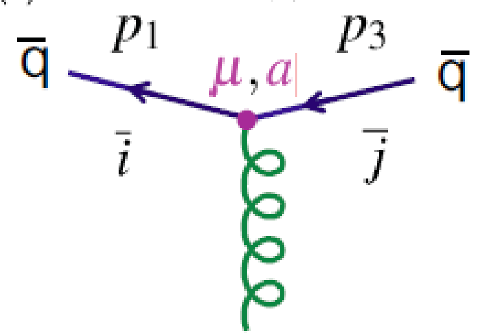
- Only matrices with non-zero entries in the 13 and 32 position
- But none of the λ matrices have non-zero entries in the 13 and 32 positions. Hence the colour factor is zero

★ colour is conserved

Colour factors: quark vs anti-quark

- Recall the colour part of wave-function: $r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$
- The QCD qqg vertex was written:

$$\bar{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_iu(p_1)$$



★ Now consider the anti-quark vertex

- The QCD q̄q̄g vertex is:

$$\bar{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_jv(p_3)$$

Note that the incoming anti-particle now enters on the LHS of the expression

- For which the colour part is

$$c_i^\dagger\lambda^a c_j = c_i^\dagger\begin{pmatrix}\lambda^a_{1j}\\ \lambda^a_{2j}\\ \lambda^a_{3j}\end{pmatrix} = \lambda^a_{ij}$$

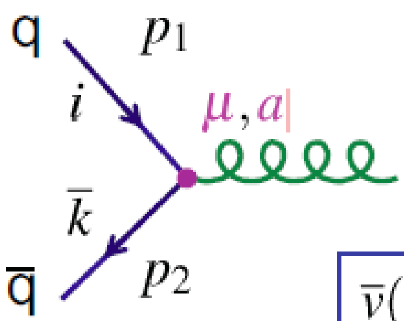
i.e indices *ij* are swapped with respect to the quark case

- Hence $\bar{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_jv(p_3) \equiv \bar{v}(p_1)\{-\tfrac{1}{2}ig_s\lambda^a_{ij}\gamma^\mu\}v(p_3)$

- c.f. the quark - gluon QCD interaction

$$\bar{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_iu(p_1) \equiv \bar{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

★ **Finally we can consider the quark – anti-quark annihilation**

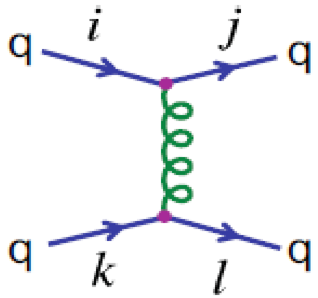


QCD vertex: $\bar{v}(p_2)c_k^\dagger\{-\frac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$

with $c_k^\dagger\lambda^a c_i = \lambda_{ki}^a$

$$\bar{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \bar{v}(p_2)\{-\tfrac{1}{2}ig_s\lambda_{ki}^a\gamma^\mu\}u(p_1)$$

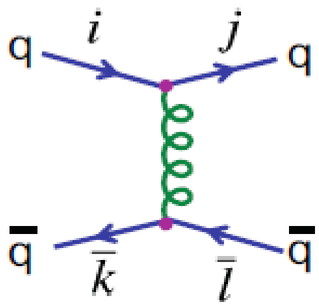
- Consequently the colour factors for the different diagrams are:



$$C(ik \rightarrow jl) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{lk}$$

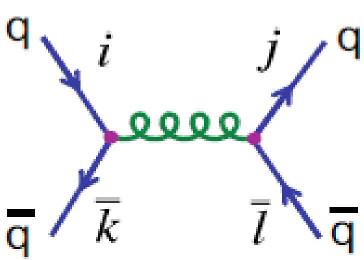
e.g.

$$C(rr \rightarrow rr) = \tfrac{1}{3}$$
$$C(rg \rightarrow rg) = -\tfrac{1}{6}$$
$$C(rg \rightarrow gr) = \tfrac{1}{2}$$



$$C(i\bar{k} \rightarrow j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{kl}$$

$$C(r\bar{r} \rightarrow r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \rightarrow r\bar{g}) = -\tfrac{1}{6}$$
$$C(r\bar{r} \rightarrow g\bar{g}) = \tfrac{1}{2}$$

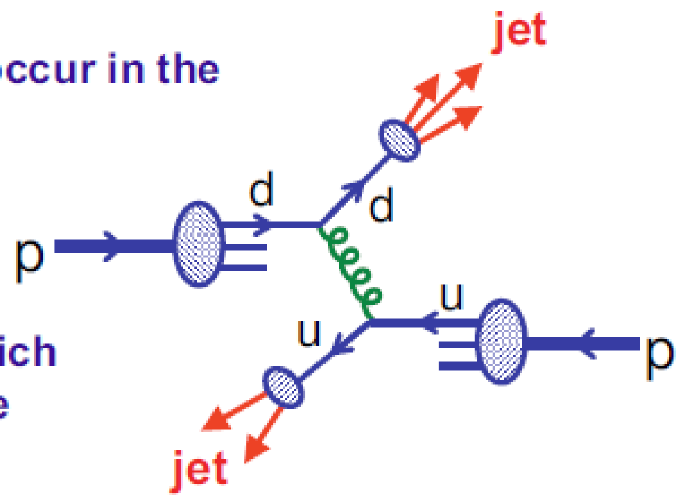


$$C(i\bar{k} \rightarrow j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ki}\lambda^a_{jl}$$

$$C(r\bar{r} \rightarrow r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \rightarrow r\bar{g}) = \tfrac{1}{2}$$
$$C(r\bar{r} \rightarrow g\bar{g}) = -\tfrac{1}{6}$$

Colour index of adjoint spinor comes first

Quark-quark scattering



- Consider the process $u+d \to u+d$ which can occur in the high energy proton-proton scattering
- There are nine possible colour configurations of the colliding quarks which are all equally likely.
- Need to determine the average matrix element which is the sum over all possible colours divided by the number of possible initial colour states

$$\langle |M_{fi}|^2 \rangle = \frac{1}{3} \cdot \frac{1}{3} \sum_{i,j,k,l=1}^{3} |M_{fi}(ij \to kl)|^2$$

- The colour average matrix element contains the average colour factor

$$\langle |C|^2 \rangle = \frac{1}{9} \sum_{i,j,k,l=1}^{3} |C(ij \to kl)|^2$$

- For $qq \to qq$

$rr \to rr,..$ $rb \to rb,..$ $rb \to br,..$

$$\langle |C|^2 \rangle = \frac{1}{9}\left[3 \times \left(\frac{1}{3}\right)^2 + 6 \times \left(-\frac{1}{6}\right)^2 + 6 \times \left(\frac{1}{2}\right)^2\right] = \frac{2}{9}$$

- Previously derived the Lorentz Invariant cross section for $e^-\mu^- \rightarrow e^-\mu^-$ elastic scattering in the ultra-relativistic limit (handout 6).

QED

$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2}{q^4}\left[1+\left(1+\frac{q^2}{s}\right)^2\right]$$

- For $ud \rightarrow ud$ in QCD replace $\alpha \longrightarrow \alpha_s$ and multiply by $\langle |C|^2 \rangle$

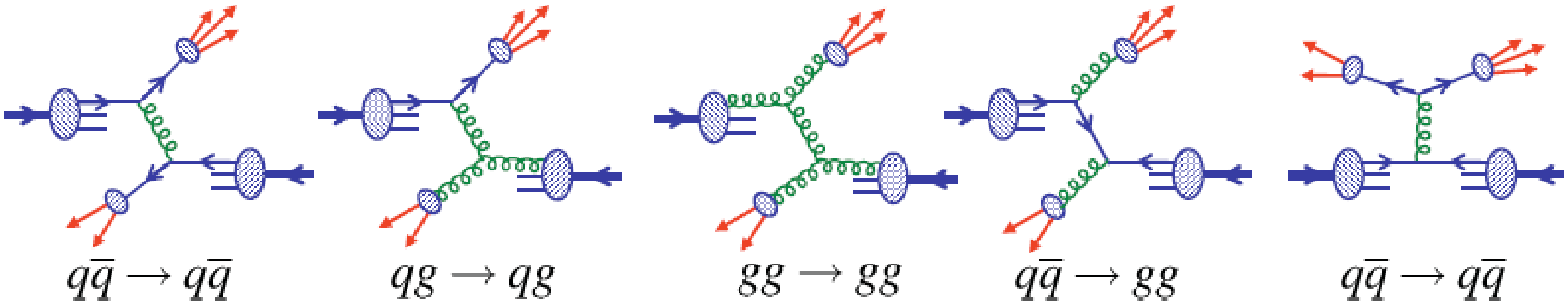
QCD

$$\frac{d\sigma}{dq^2} = \frac{2}{9}\frac{2\pi\alpha_S^2}{q^4}\left[1+\left(1+\frac{q^2}{\hat{s}}\right)^2\right]$$

Never see colour, but enters through colour factors. Can tell QCD is SU(3)

- Here $\hat{s}$ is the centre-of-mass energy of the quark-quark collision
- The calculation of hadron-hadron scattering is very involved, need to include parton structure functions and include all possible interactions

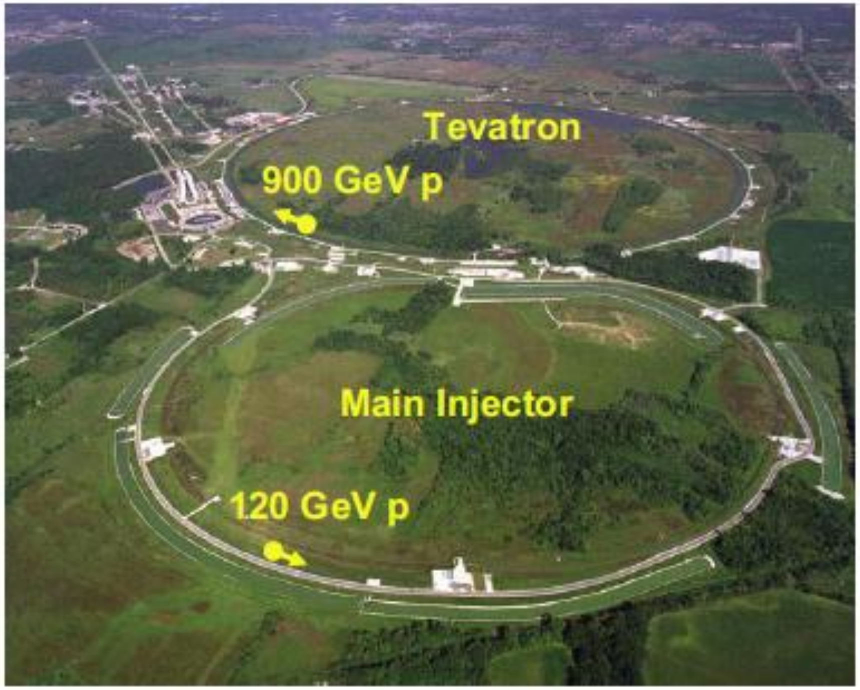
e.g. two jet production in proton-antiproton collisions



Proton-antiproton collisions at Tevatron

- ★ Tevatron collider at Fermi National Laboratory (FNAL)
  - located ~40 miles from Chigaco, US
  - started operation in 1987 (will run until 2009/2010)
- ★ $p\bar{p}$ collisions at $\sqrt{s}$ = 1.8 TeV

c.f. 14 TeV at the LHC

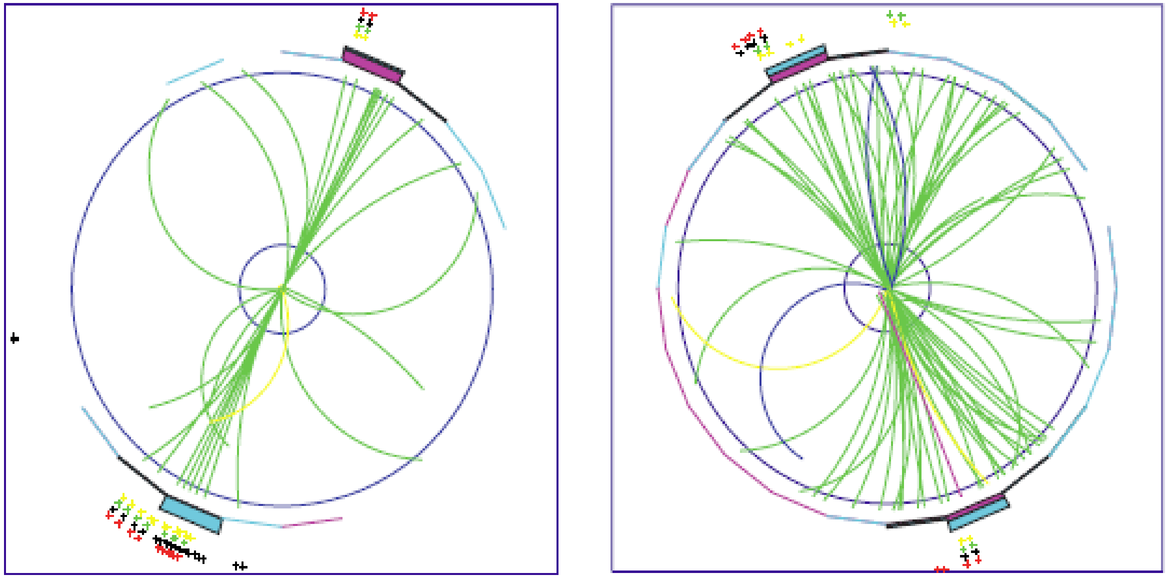


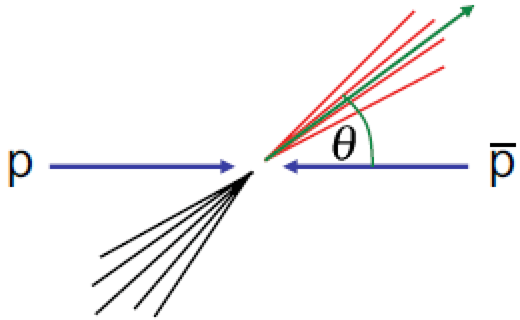
Two main accelerators:

- ★ Main Injector
  - Accelerates 8 GeV p to 120 GeV
  - also $\overline{p}$ to 120 GeV
  - Protons sent to Tevatron & MINOS
  - $\overline{p}$ all go to Tevatron
- ★ Tevatron
  - 4 mile circumference
  - accelerates $p/\overline{p}$ from 120 GeV to 900 GeV

★ Test QCD predictions by looking at production of pairs of high energy jets

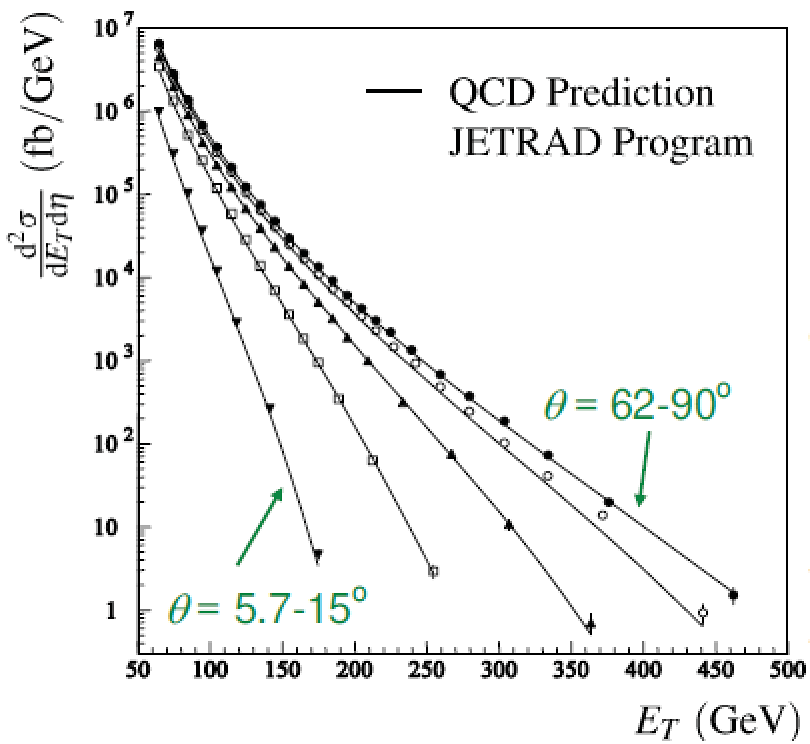
$p\bar{p} \rightarrow$ jet jet + X





- ★ **Measure cross-section in terms of**
  - "transverse energy" $E_T = E_{\text{jet}} \sin\theta$
  - "pseudorapidity" $\eta = \ln\left[\cot\left(\frac{\theta}{2}\right)\right]$

…don't worry too much about the details here, what matters is that…



D0 Collaboration, Phys. Rev. Lett. 86 (2001)

> ★ **QCD predictions provide an excellent description of the data**

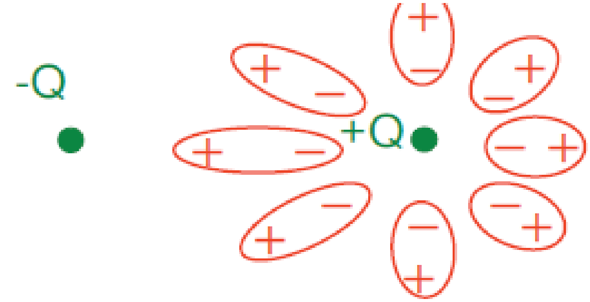
★ **NOTE:**

- at low E_T cross-section is dominated by low x partons
  - i.e. gluon-gluon scattering
- at high E_T cross-section is dominated by high x partons
  - i.e. quark-antiquark scattering

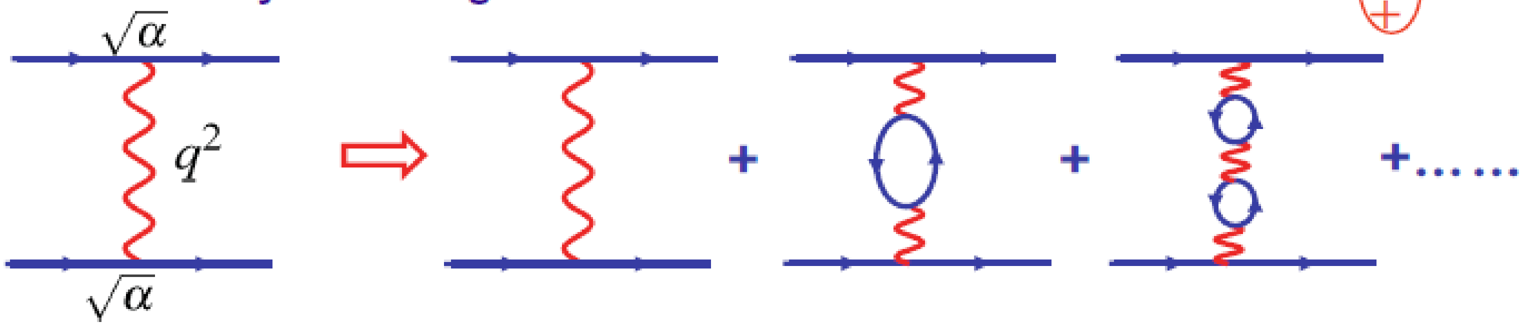
Running coupling constants

QED

- "bare" charge of electron screened by virtual e^+e^- pairs
- behaves like a polarizable dielectric

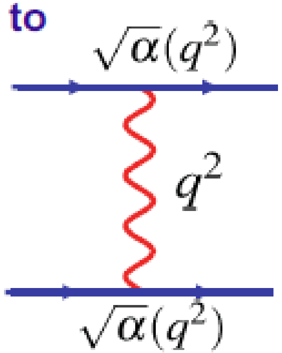


★ In terms of Feynman diagrams:



★ Same final state so add matrix element amplitudes: $M = M_1 + M_2 + M_3 + \ldots$

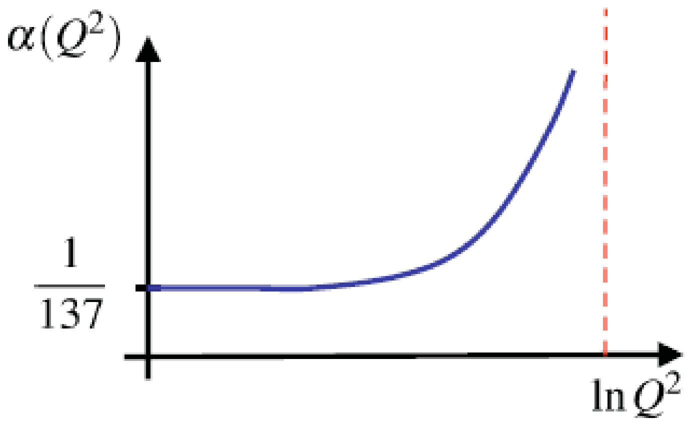
★ Giving an infinite series which can be summed and is equivalent to a single diagram with "running" coupling constant

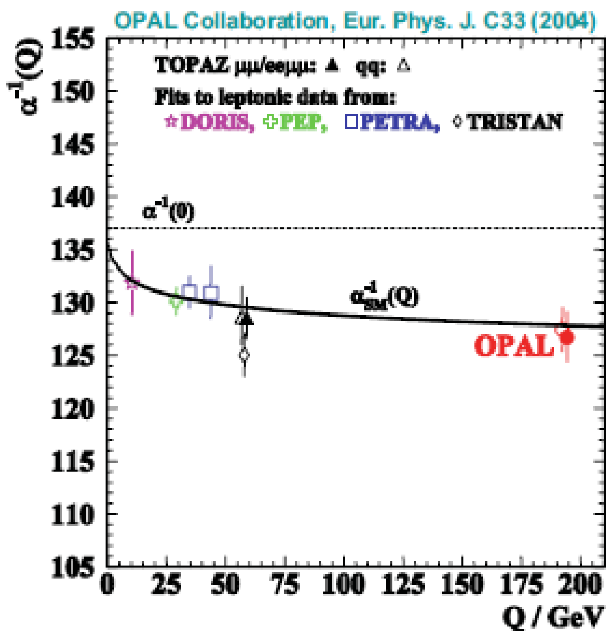


$$\alpha(Q^2) = \alpha(Q_0^2) \Big/ \left[1 - \frac{\alpha(Q_0^2)}{3\pi} \ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

$$Q^2 \gg Q_0^2$$

Note sign





- ★ Might worry that coupling becomes infinite at

$$\ln\left(\frac{Q^2}{Q_0^2}\right) = \frac{3\pi}{1/137}$$

  i.e. at $Q \sim 10^{26}\,\mathrm{GeV}$

  - But quantum gravity effects would come in way below this energy and it is highly unlikely that QED "as is" would be valid in this regime

- ★ In QED, running coupling increases very slowly
  - Atomic physics: $Q^2 \sim 0$

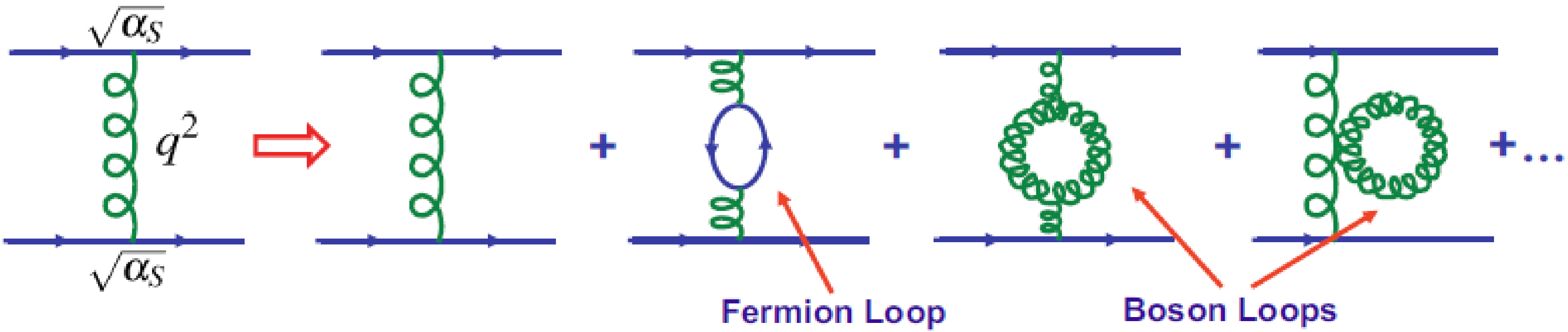
$$1/\alpha = 137.03599976(50)$$

  - High energy physics:

$$1/\alpha(193\,\mathrm{GeV}) = 127.4 \pm 2.1$$

Running of α_s

QCD Similar to QED but also have gluon loops



★ Remembering **adding amplitudes**, so can get negative interference and the sum can be smaller than the original diagram alone

★ Bosonic loops "interfere negatively"

$$\alpha_S(Q^2) = \alpha_S(Q_0^2) \Big/ \left[1 + B\alpha_S(Q_0^2)\ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

with $$B = \frac{11N_c - 2N_f}{12\pi}$$

N_c = no. of colours

N_f = no. of quark flavours

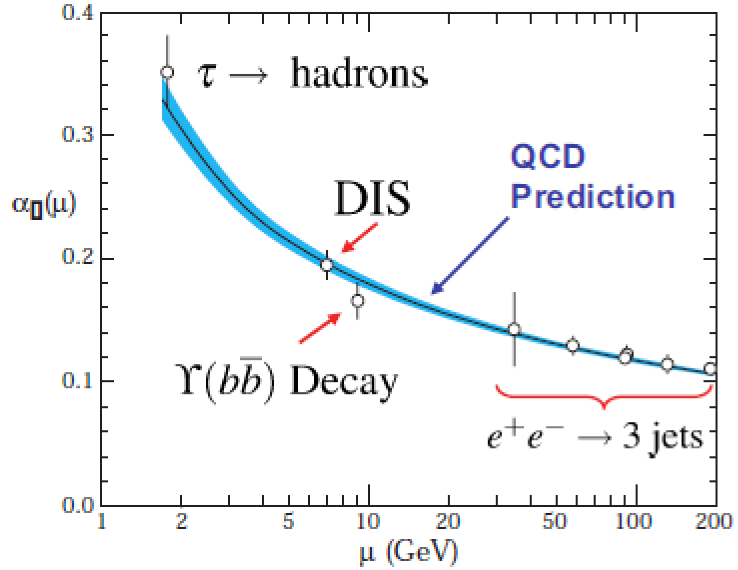
$N_c = 3;\ N_f = 6$ ➡ $B > 0$

➡ **α_S decreases with Q^2**

Nobel Prize for Physics, 2004
(Gross, Politzer, Wilczek)

★ Measure α_s in many ways:
- jet rates
- DIS
- tau decays
- bottomonium decays
- +...

★ As predicted by QCD, α_s decreases with Q^2



$\mu \approx Q$

★ At low Q^2: α_s is large, e.g. at $Q^2 = 1\,\mathrm{GeV}^2$ find $\alpha_s \sim 1$

- Can't use perturbation theory ! This is the reason why QCD calculations at low energies are so difficult, e.g. properties hadrons, hadronisation of quarks to jets,...

★ At high Q^2 : α_s is rather small, e.g. at $Q^2 = M_Z^2$ find $\alpha_s \sim 0.12$

➡ **Asymptotic Freedom**

- Can use perturbation theory and this is the reason that in DIS at high Q^2 quarks behave as if they are quasi-free (i.e. only weakly bound within hadrons)

Summary

★ Superficially QCD very similar to QED

★ But gluon self-interactions are believed to result in colour confinement

★ All hadrons are colour singlets which explains why only observe

Mesons **Baryons**

★ A low energies $\alpha_S \sim 1$

➡ Can't use perturbation theory !

Non-Perturbative regime

★ Coupling constant runs, smaller coupling at higher energy scales

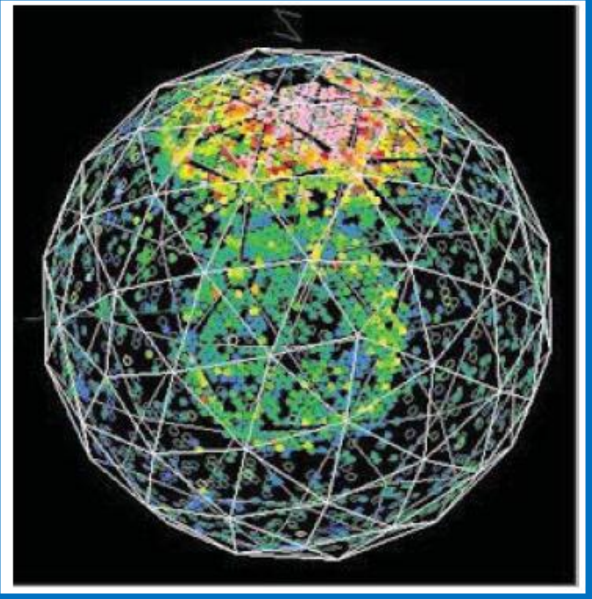
$$\alpha_S(100\,\mathrm{GeV}) \sim 0.1$$

➡ Can use perturbation theory

Asymptotic Freedom

★ Where calculations can be performed, QCD provides a good description of relevant experimental data

The weak interaction and V-A



Parity

★The parity operator performs spatial inversion through the origin:

$$\psi'(\vec{x},t) = \hat{P}\psi(\vec{x},t) = \psi(-\vec{x},t)$$

• applying $\hat{P}$ twice: $\hat{P}\hat{P}\psi(\vec{x},t) = \hat{P}\psi(-\vec{x},t) = \psi(\vec{x},t)$

so $\hat{P}\hat{P} = I$ ➡ $\hat{P}^{-1} = \hat{P}$

• To preserve the normalisation of the wave-function

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\psi|\hat{P}^{\dagger}\hat{P}|\psi\rangle$$

$\hat{P}^{\dagger}\hat{P} = I$ ➡ $\hat{P}$ Unitary

• But since $\hat{P}\hat{P} = I$ $\hat{P} = \hat{P}^{\dagger}$ ➡ $\hat{P}$ Hermitian

which implies Parity is an observable quantity. If the interaction Hamiltonian commutes with $\hat{P}$, parity is an observable conserved quantity

• If $\psi(\vec{x},t)$ is an eigenfunction of the parity operator with eigenvalue P

$\hat{P}\psi(\vec{x},t) = P\psi(\vec{x},t)$ ➡ $\hat{P}\hat{P}\psi(\vec{x},t) = P\hat{P}\psi(\vec{x},t) = P^2\psi(\vec{x},t)$

since $\hat{P}\hat{P} = I$ $P^2 = 1$

➡ Parity has eigenvalues $P = \pm 1$

★ QED and QCD are invariant under parity

★ Experimentally observe that Weak Interactions do not conserve parity

Intrinsic Parities of fundamental particles:

Spin-1 Bosons

- From Gauge Field Theory can show that the gauge bosons have $P = -1$

$$P_\gamma = P_g = P_{W^+} = P_{W^-} = P_Z = -1$$

Spin-½ Fermions

- From the Dirac equation showed (handout 2):

  Spin ½ particles have opposite parity to spin ½ anti-particles

- Conventional choice: spin ½ particles have $P = +1$

$$P_{e^-} = P_{\mu^-} = P_{\tau^-} = P_\nu = P_q = +1$$

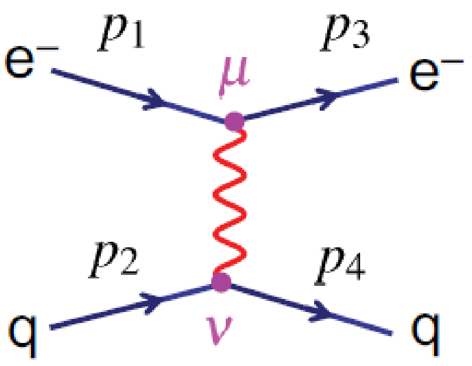
and anti-particles have opposite parity, i.e.

$$P_{e^+} = P_{\mu^+} = P_{\tau^+} = P_{\overline{\nu}} = P_{\overline{q}} = -1$$

★ For Dirac spinors it was shown (handout 2) that the parity operator is:

$$\hat{P} = \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Parity conservation in QED and QCD



- Consider the QED process e⁻q → e⁻q
- The Feynman rules for QED give:

$$iM = [\overline{u}_e(p_3)ie\gamma^\mu u_e(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}_q(p_4)ie\gamma^\nu u_q(p_2)]$$

- Which can be expressed in terms of the electron and quark 4-vector currents:

$$M = -\frac{e^2}{q^2}g_{\mu\nu}j_e^\mu j_q^\nu = -\frac{e^2}{q^2}j_e . j_q$$

with $j_e = \overline{u}_e(p_3)\gamma^\mu u_e(p_1)$ and $j_q = \overline{u}_q(p_4)\gamma^\mu u_q(p_2)$

★ Consider the what happen to the matrix element under the parity transformation

- Spinors transform as $u \xrightarrow{\hat{P}} \hat{P}u = \gamma^0 u$
- Adjoint spinors transform as

$$\overline{u} = u^\dagger\gamma^0 \xrightarrow{\hat{P}} (\hat{P}u)^\dagger\gamma^0 = u^\dagger\gamma^{0\dagger}\gamma^0 = u^\dagger\gamma^0\gamma^0 = \overline{u}\gamma^0$$

$$\overline{u} \xrightarrow{\hat{P}} \overline{u}\gamma^0$$

- Hence $j_e = \overline{u}_e(p_3)\gamma^\mu u_e(p_1) \xrightarrow{\hat{P}} \overline{u}_e(p_3)\gamma^0\gamma^\mu\gamma^0 u_e(p_1)$

★ Consider the components of the four-vector current

0: $j_e^0 \xrightarrow{\hat{P}} \overline{u}\gamma^0\gamma^0\gamma^0 u = \overline{u}\gamma^0 u = j_e^0$ since $\gamma^0\gamma^0 = 1$

k=1,2,3: $j_e^k \xrightarrow{\hat{P}} \overline{u}\gamma^0\gamma^k\gamma^0 u = -\overline{u}\gamma^k\gamma^0\gamma^0 u = -\overline{u}\gamma^k u = -j_e^k$ since $\gamma^0\gamma^k = -\gamma^k\gamma^0$

- The time-like component remains unchanged and the space-like components change sign
- Similarly $j_q^0 \xrightarrow{\hat{P}} j_q^0$ $j_q^k \xrightarrow{\hat{P}} -j_q^k$ $k = 1,2,3$

or $j^\mu \xrightarrow{\hat{P}} j_\mu$

$j^\mu . j^\nu \xrightarrow{\hat{P}} j_\mu . j_\nu \xrightarrow{\hat{P}} j^\mu . j^\nu$

★ Consequently the four-vector scalar product

$$j_e . j_q = j_e^0 j_q^0 - j_e^k j_q^k \xrightarrow{\hat{P}} j_e^0 j_q^0 - (-j_e^k)(-j_q^k) = j_e . j_q \quad k = 1,3$$

QED Matrix Elements are Parity Invariant

➡ **Parity Conserved in QED**

★ The QCD vertex has the same form and, thus,

Parity Conserved in QCD

Parity violation in β-decay

★The parity operator $\hat{P}$ corresponds to a discrete transformation $x \rightarrow -x, etc.$

★Under the parity transformation:

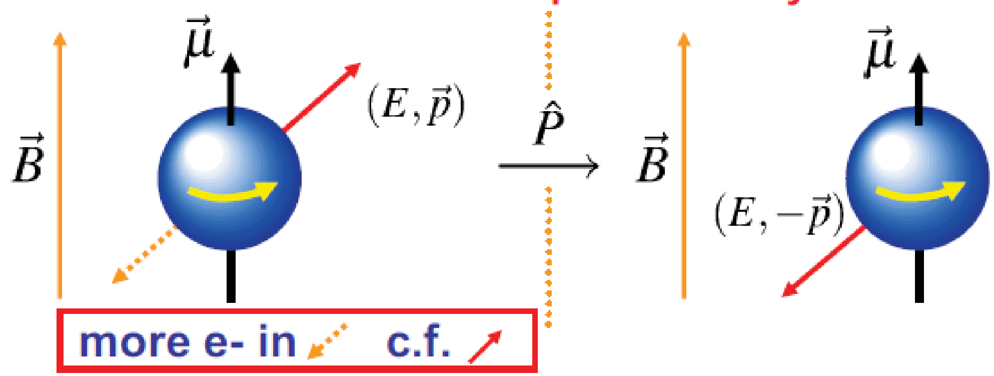
Vectors change sign
$$\vec{r} \xrightarrow{\hat{P}} -\vec{r}$$
$$\vec{p} \xrightarrow{\hat{P}} -\vec{p} \qquad (p_x = \frac{\partial}{\partial x}, etc.)$$

Axial-Vectors unchanged
$$\vec{L} \xrightarrow{\hat{P}} \vec{L} \qquad (\vec{L} = \vec{r} \wedge \vec{p})$$
$$\vec{\mu} \xrightarrow{\hat{P}} \vec{\mu} \qquad (\vec{\mu} \propto \vec{L})$$

Note B is an axial vector
$d\vec{B} \propto \vec{J} \wedge \vec{r} d^3\vec{r}$

★**1957:** C.S.Wu et al. studied beta decay of polarized cobalt-60 nuclei:
$$^{60}\mathrm{Co} \rightarrow ^{60}Ni^* + e^- + \overline{\nu}_e$$

★Observed electrons emitted preferentially in direction opposite to applied field



If parity were conserved: expect equal rate for producing e⁻ in directions along and opposite to the nuclear spin.

★Conclude parity is violated in WEAK INTERACTION
→ that the WEAK interaction vertex is NOT of the form $\overline{u}_e \gamma^{\mu} u_{\nu}$

Bilinear covariance

★ The requirement of Lorentz invariance of the matrix element severely restricts the form of the interaction vertex. QED and QCD are "VECTOR" interactions:

$$j^{\mu} = \overline{\psi}\gamma^{\mu}\phi$$

★ This combination transforms as a 4-vector (Handout 2 appendix V)

★ In general, there are only 5 possible combinations of two spinors and the gamma matrices that form Lorentz invariant currents, called "bilinear covariants":

Type	Form	Components	"Boson Spin"
♦ SCALAR	$\overline{\psi}\phi$	1	0
♦ PSEUDOSCALAR	$\overline{\psi}\gamma^{5}\phi$	1	0
♦ VECTOR	$\overline{\psi}\gamma^{\mu}\phi$	4	1
♦ AXIAL VECTOR	$\overline{\psi}\gamma^{\mu}\gamma^{5}\phi$	4	1
♦ TENSOR	$\overline{\psi}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})\phi$	6	2

★ Note that in total the sixteen components correspond to the 16 elements of a general 4x4 matrix: "decomposition into Lorentz invariant combinations"

★ In QED the factor $g_{\mu\nu}$ arose from the sum over polarization states of the virtual photon (2 transverse + 1 longitudinal, 1 scalar) = (2J+1) + 1

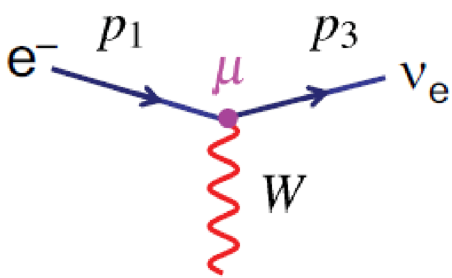
★ Associate SCALAR and PSEUDOSCALAR interactions with the exchange of a SPIN-0 boson, etc. – no spin degrees of freedom

V-A structure of weak interaction

★ The most general form for the interaction between a fermion and a boson is a linear combination of bilinear covariants

★ For an interaction corresponding to the **exchange of a spin-1** particle the most general form is a linear combination of VECTOR and AXIAL-VECTOR

★ The form for WEAK interaction is <u>determined from experiment</u> to be VECTOR – AXIAL-VECTOR (V – A)



$$j^{\mu} \propto \overline{u}_{\nu_e}(\gamma^{\mu} - \gamma^{\mu}\gamma^{5})u_e$$

V – A

★ Can this account for parity violation?

★ First consider parity transformation of a **pure AXIAL-VECTOR** current

$$j_A = \overline{\psi}\gamma^{\mu}\gamma^{5}\phi \qquad \text{with} \qquad \gamma^{5} = i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}; \quad \gamma^{5}\gamma^{0} = -\gamma^{0}\gamma^{5}$$

$$j_A = \overline{\psi}\gamma^{\mu}\gamma^{5}\phi \xrightarrow{\hat{P}} \overline{\psi}\gamma^{0}\gamma^{\mu}\gamma^{5}\gamma^{0}\phi = -\overline{\psi}\gamma^{0}\gamma^{\mu}\gamma^{0}\gamma^{5}\phi$$

$$j_A^{0} = \xrightarrow{\hat{P}} -\overline{\psi}\gamma^{0}\gamma^{0}\gamma^{0}\gamma^{5}\phi = -\overline{\psi}\gamma^{0}\gamma^{5}\phi = -j_A^{0}$$

$$j_A^{k} = \xrightarrow{\hat{P}} -\overline{\psi}\gamma^{0}\gamma^{k}\gamma^{0}\gamma^{5}\phi = +\overline{\psi}\gamma^{k}\gamma^{5}\phi = +j_A^{k} \qquad k = 1,2,3$$

or $j_A^{\mu} \xrightarrow{\hat{P}} -j_{A\mu}$

- The space-like components remain unchanged and the time-like components change sign (the opposite to the parity properties of a vector-current)

$$j_A^0 \xrightarrow{\hat{P}} -j_A^0; \quad j_A^k \xrightarrow{\hat{P}} +j_A^k; \qquad j_V^0 \xrightarrow{\hat{P}} +j_V^0; \quad j_V^k \xrightarrow{\hat{P}} -j_V^k$$

- Now consider the matrix elements

$$M \propto g_{\mu\nu} j_1^\mu j_2^\nu = j_1^0 j_2^0 - \sum_{k=1,3} j_1^k j_2^k$$

- For the combination of a two axial-vector currents

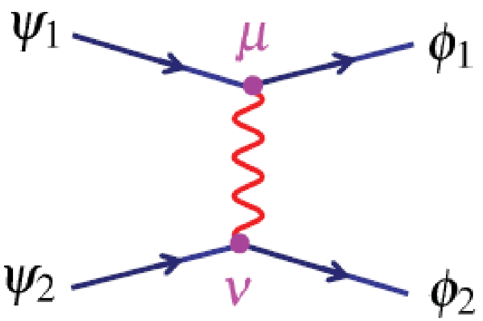
$$j_{A1} \cdot j_{A2} \xrightarrow{\hat{P}} (-j_1^0)(-j_2^0) - \sum_{k=1,3} (j_1^k)(j_2^k) = j_{A1} \cdot j_{A2}$$

- Consequently parity is conserved for both a pure vector and pure axial-vector interactions
- However the combination of a vector current and an axial vector current

$$j_{V1} \cdot j_{A2} \xrightarrow{\hat{P}} (j_1^0)(-j_2^0) - \sum_{k=1,3} (-j_1^k)(j_2^k) = -j_{V1} \cdot j_{A2}$$

changes sign under parity – can give parity violation !

★ Now consider a general linear combination of VECTOR and AXIAL-VECTOR (note this is relevant for the Z-boson vertex)



$$j_1 = \overline{\phi}_1(g_V\gamma^\mu + g_A\gamma^\mu\gamma^5)\psi_1 = g_V j_1^V + g_A j_1^A$$

$$\frac{g_{\mu\nu}}{q^2 - m^2}$$

$$j_2 = \overline{\phi}_2(g_V\gamma^\mu + g_A\gamma^\mu\gamma^5)\psi_2 = g_V j_2^V + g_A j_2^A$$

$$M_{fi} \propto j_1.j_2 = g_V^2 j_1^V.j_2^V + g_A^2 j_1^A.j_2^A + g_V g_A(j_1^V.j_2^A + j_1^A.j_2^V)$$

- Consider the parity transformation of this scalar product

$$j_1.j_2 \xrightarrow{\hat{P}} g_V^2 j_1^V.j_2^V + g_A^2 j_1^A.j_2^A - g_V g_A(j_1^V.j_2^A + j_1^A.j_2^V)$$

- If either g_A or g_V is zero, Parity is conserved, i.e. parity conserved in a pure VECTOR or pure AXIAL-VECTOR interaction

- Relative strength of parity violating part $\propto \dfrac{g_V g_A}{g_V^2 + g_A^2}$

Maximal Parity Violation for V-A (or V+A)

Chiral structure of QED (reminder)

★ Recall introduced CHIRAL projections operators

$$P_R = \tfrac{1}{2}(1+\gamma^5); \qquad P_L = \tfrac{1}{2}(1-\gamma^5)$$

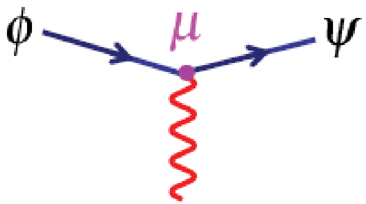
project out chiral right- and left- handed states

★ In the ultra-relativistic limit, chiral states correspond to helicity states

★ Any spinor can be expressed as:

$$\psi = \tfrac{1}{2}(1+\gamma^5)\psi + \tfrac{1}{2}(1-\gamma^5)\psi = P_R\psi + P_L\psi = \psi_R + \psi_L$$

• The QED vertex $\overline{\psi}\gamma^\mu\phi$ in terms of chiral states:

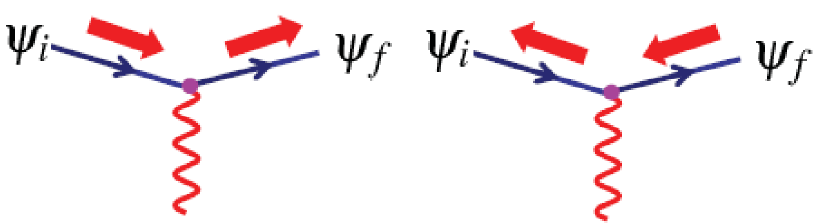


$$\overline{\psi}\gamma^\mu\phi = \overline{\psi}_R\gamma^\mu\phi_R + \overline{\psi}_R\gamma^\mu\phi_L + \overline{\psi}_L\gamma^\mu\phi_R + \overline{\psi}_L\gamma^\mu\phi_L$$

conserves chirality, e.g.

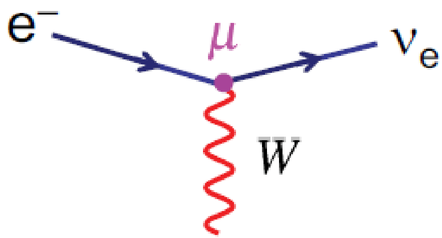
$$\begin{aligned}
\overline{\psi}_R\gamma^\mu\phi_L &= \tfrac{1}{2}\psi^\dagger(1+\gamma^5)\gamma^0\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\phi \\
&= \tfrac{1}{4}\psi^\dagger\gamma^0(1-\gamma^5)\gamma^\mu(1-\gamma^5)\phi \\
&= \tfrac{1}{4}\overline{\psi}\gamma^\mu(1+\gamma^5)(1-\gamma^5)\phi = 0
\end{aligned}$$

★ In the ultra-relativistic limit only two helicity combinations are non-zero



Helicity structure of weak interactions

★The charged current ($W^{\pm}$) weak vertex is:



$$\boxed{\frac{-ig_W}{\sqrt{2}}\frac{1}{2}\gamma^{\mu}(1-\gamma^5)}$$

★Since $\frac{1}{2}(1-\gamma^5)$ projects out left-handed chiral particle states:

$$\overline{\psi}\tfrac{1}{2}\gamma^{\mu}(1-\gamma^5)\phi = \overline{\psi}\gamma^{\mu}\phi_L$$

★Writing $\overline{\psi} = \overline{\psi}_R + \overline{\psi}_L$ and from discussion of QED, $\overline{\psi}_R\gamma^{\mu}\phi_L = 0$ gives

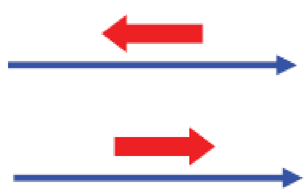
$$\overline{\psi}\tfrac{1}{2}\gamma^{\mu}(1-\gamma^5)\phi = \overline{\psi}_L\gamma^{\mu}\phi_L$$

⇒ Only the left-handed chiral components of particle spinors and right-handed chiral components of anti-particle spinors participate in charged current weak interactions

★At very high energy $(E \gg m)$, the left-handed chiral components are helicity eigenstates :

$\frac{1}{2}(1-\gamma^5)u \Rightarrow$

$\frac{1}{2}(1-\gamma^5)v \Rightarrow$

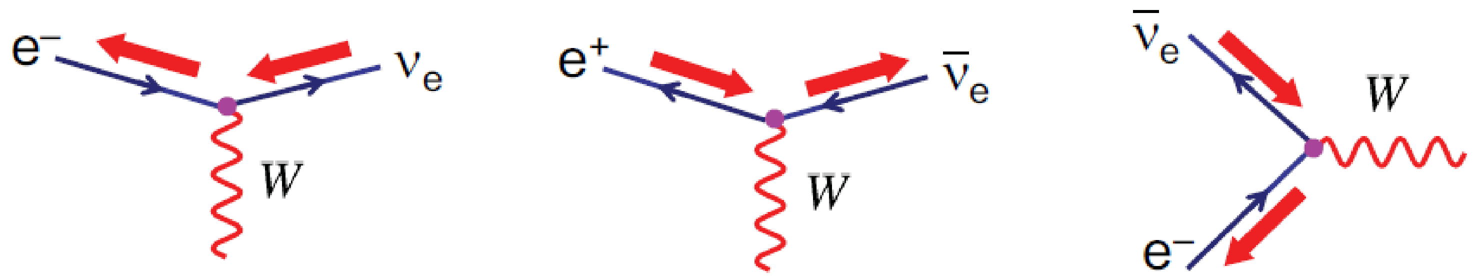


LEFT-HANDED PARTICLES
Helicity = -1

RIGHT-HANDED ANTI-PARTICLES
Helicity = +1

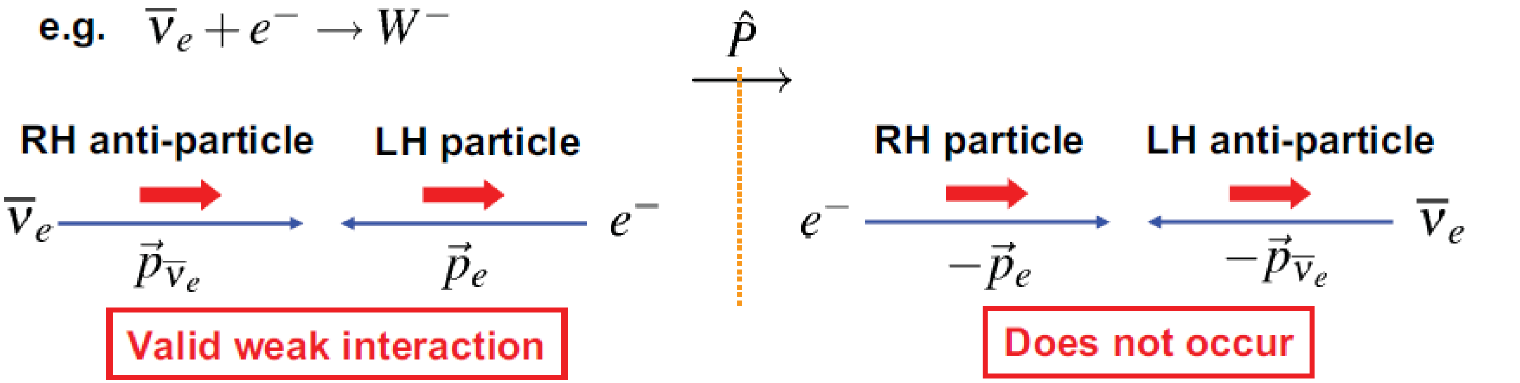
In the ultra-relativistic limit only left-handed particles and right-handed antiparticles participate in charged current weak interactions

e.g. In the relativistic limit, the only possible electron – neutrino interactions are:



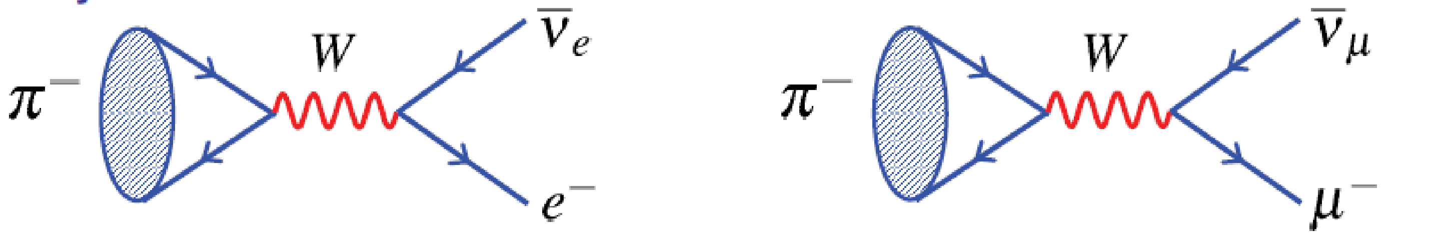
★ The helicity dependence of the weak interaction ⟷ parity violation

e.g. $\bar{\nu}_e + e^- \rightarrow W^-$



Helicity in pion decay

★ The decays of charged pions provide a good demonstration of the role of helicity in the weak interaction

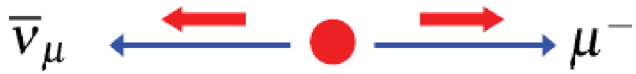


EXPERIMENTALLY: $$\frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)} = 1.23 \times 10^{-4}$$

- Might expect the decay to electrons to dominate – due to increased phase space.... The opposite happens, the electron decay is helicity suppressed

★ Consider decay in pion rest frame.

- Pion is spin zero: so the spins of the $\overline{\nu}$ and μ are opposite
- Weak interaction only couples to RH chiral anti-particle states. Since neutrinos are (almost) massless, must be in RH Helicity state
- Therefore, to conserve angular mom. muon is emitted in a RH **HELICITY** state



- But only left-handed **CHIRAL** particle states participate in weak interaction

★The general right-handed helicity solution to the Dirac equation is

$$u_\uparrow = N \begin{pmatrix} c \\ e^{i\phi} s \\ \frac{|\vec{p}|}{E+m} c \\ \frac{|\vec{p}|}{E+m} e^{i\phi} s \end{pmatrix} \quad \text{with} \quad c = \cos\frac{\theta}{2} \quad \text{and} \quad s = \sin\frac{\theta}{2}$$

- project out the left-handed chiral part of the wave-function using

$$P_L = \tfrac{1}{2}(1-\gamma^5) = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

giving

$$P_L u_\uparrow = \tfrac{1}{2}N\left(1-\frac{|\vec{p}|}{E+m}\right)\begin{pmatrix} c \\ e^{i\phi} s \\ -c \\ -e^{i\phi} s \end{pmatrix} = \tfrac{1}{2}N\left(1-\frac{|\vec{p}|}{E+m}\right) u_L$$

In the limit $m \ll E$ this tends to zero

- similarly

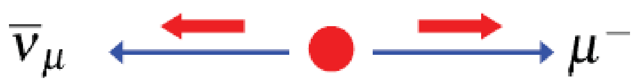
$$P_R u_\uparrow = \tfrac{1}{2}N\left(1+\frac{|\vec{p}|}{E+m}\right)\begin{pmatrix} c \\ e^{i\phi} s \\ c \\ e^{i\phi} s \end{pmatrix} = \tfrac{1}{2}N\left(1+\frac{|\vec{p}|}{E+m}\right) u_R$$

In the limit $m \ll E$, $P_R u_\uparrow \rightarrow u_R$

★ Hence $$u_{\uparrow} = P_R u_{\uparrow} + P_L u_{\uparrow} = \frac{1}{2}\left(1 + \frac{|\vec{p}|}{E+m}\right) u_R + \frac{1}{2}\left(1 - \frac{|\vec{p}|}{E+m}\right) u_L$$

RH Helicity ($u_{\uparrow}$) — RH Chiral (u_R) — LH Chiral (u_L)

- In the limit $E \gg m$, as expected, the RH chiral and helicity states are identical
- Although only LH chiral particles participate in the weak interaction the contribution from RH **Helicity** states is not necessarily zero !



$m_\nu \approx 0$: RH Helicity ≡ RH Chiral

$m_\mu \neq 0$: RH Helicity has LH Chiral Component

★ Expect matrix element to be proportional to LH chiral component of RH Helicity electron/muon spinor

$$M_{fi} \propto \frac{1}{2}\left(1 - \frac{|\vec{p}|}{E+m}\right) = \frac{m_\mu}{m_\pi + m_\mu}$$

from the kinematics of pion decay at rest

★ Hence because the electron mass is much smaller than the pion mass the decay $\pi^- \rightarrow e^- \overline{\nu}_e$ is heavily suppressed.

Evidence for V-A

★ The V-A nature of the charged current weak interaction vertex fits with experiment

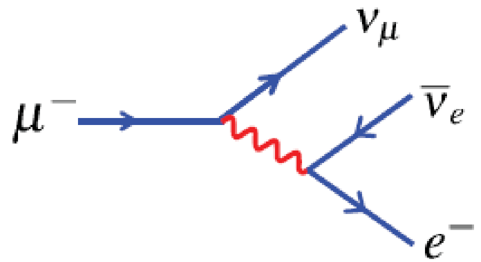
EXAMPLE charged pion decay

- Experimentally measure: $\dfrac{\Gamma(\pi^- \to e^- \overline{\nu}_e)}{\Gamma(\pi^- \to \mu^- \overline{\nu}_\mu)} = (1.230 \pm 0.004) \times 10^{-4}$
- Theoretical predictions (depend on Lorentz Structure of the interaction)

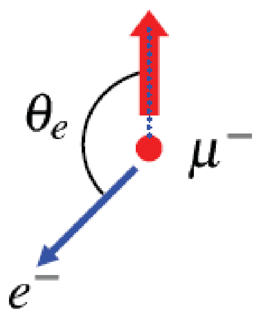
V-A $(\overline{\psi}\gamma^\mu(1-\gamma^5)\phi)$ or V+A $(\overline{\psi}\gamma^\mu(1+\gamma^5)\phi)$ ➡ $\dfrac{\Gamma(\pi^- \to e^- \overline{\nu}_e)}{\Gamma(\pi^- \to \mu^- \overline{\nu}_\mu)} \approx 1.3 \times 10^{-4}$

Scalar $(\overline{\psi}\phi)$ or Pseudo-Scalar $(\overline{\psi}\gamma^5\phi)$ ➡ $\dfrac{\Gamma(\pi^- \to e^- \overline{\nu}_e)}{\Gamma(\pi^- \to \mu^- \overline{\nu}_\mu)} = 5.5$

EXAMPLE muon decay



Measure electron energy and angular distributions relative to muon spin direction. Results expressed in terms of general S+P+V+A+T form in "Michel Parameters"



e.g. TWIST expt: 6x10⁹ μ decays
Phys. Rev. Lett. 95 (2005) 101805

$$\rho = 0.75080 \pm 0.00105$$

V-A Prediction: $\rho = 0.75$

Weak charged current propagator

★ The charged-current Weak interaction is different from QED and QCD in that it is mediated by massive W-bosons (80.3 GeV)

★ This results in a more complicated form for the propagator:

- in handout 4 showed that for the exchange of a massive particle:

massless → massive

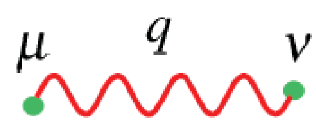
$$\frac{1}{q^2} \longrightarrow \frac{1}{q^2 - m^2}$$

- In addition the sum over W boson polarization states modifies the numerator

W-boson propagator

spin 1 $W^{\pm}$

$$\frac{-i\left[g_{\mu\nu} - q_\mu q_\nu / m_W^2\right]}{q^2 - m_W^2}$$



★ However in the limit where q^2 is small compared with $m_W = 80.3\,\mathrm{GeV}$ the interaction takes a simpler form.

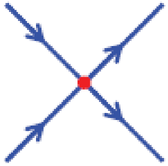
W-boson propagator ($q^2 \ll m_W^2$)

$$\frac{i g_{\mu\nu}}{m_W^2}$$

- The interaction appears point-like (i.e no q^2 dependence)

Connection to Fermi theory

★ In 1934, before the discovery of parity violation, Fermi proposed, in analogy with QED, that the invariant matrix element for β-decay was of the form:



$$M_{fi} = G_F\, g_{\mu\nu}[\overline{\psi}\gamma^{\mu}\psi][\overline{\psi}\gamma^{\nu}\psi]$$

where $G_F = 1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}$

- Note the absence of a propagator : i.e. this represents an interaction at a point

★ After the discovery of parity violation in 1957 this was modified to

$$M_{fi} = \frac{G_F}{\sqrt{2}} g_{\mu\nu}[\overline{\psi}\gamma^{\mu}(1-\gamma^5)\psi][\overline{\psi}\gamma^{\nu}(1-\gamma^5)\psi]$$

(the factor of √2 was included so the numerical value of G_F did not need to be changed)

★ Compare to the prediction for W-boson exchange

$$M_{fi} = [\tfrac{g_W}{\sqrt{2}}\overline{\psi}\tfrac{1}{2}\gamma^{\mu}(1-\gamma^5)\psi]\frac{g_{\mu\nu}-q_{\mu}q_{\nu}/m_W^2}{q^2-m_W^2}[\tfrac{g_W}{\sqrt{2}}\overline{\psi}\tfrac{1}{2}\gamma^{\nu}(1-\gamma^5)\psi]$$

which for $q^2 \ll m_W^2$ becomes:

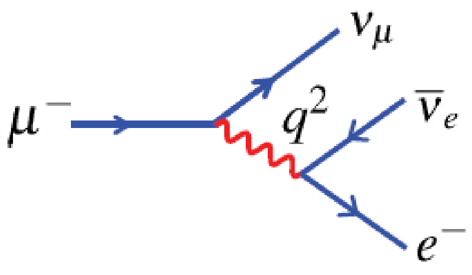
$$M_{fi} = \frac{g_W^2}{8m_W^2} g_{\mu\nu}[\overline{\psi}\gamma^{\mu}(1-\gamma^5)\psi][\overline{\psi}\gamma^{\nu}(1-\gamma^5)\psi]$$

$$\Rightarrow \boxed{\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2}}$$

Still usually use G_F to express strength of weak interaction as the is the quantity that is precisely determined in muon decay

Strength of Weak Interaction

★ Strength of weak interaction most precisely measured in muon decay



- Here $q^2 < m_\mu$ (0.106 GeV)
- To a very good approximation the W-boson propagator can be written

$$\frac{-i\left[g_{\mu\nu} - q_\mu q_\nu / m_W^2\right]}{q^2 - m_W^2} \approx \frac{i g_{\mu\nu}}{m_W^2}$$

- In muon decay measure g_W^2 / m_W^2
- Muon decay ➡ $G_F = 1.16639(1) \times 10^{-5}\,\mathrm{GeV}^{-2}$

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2}$$

★ To obtain the intrinsic strength of weak interaction need to know mass of W-boson: $m_W = 80.403 \pm 0.029\,\mathrm{GeV}$ (see handout 14)

➡ $$\alpha_W = \frac{g_W^2}{4\pi} = \frac{8m_W^2 G_F}{4\sqrt{2}\pi} = \frac{1}{30}$$

★ The intrinsic strength of the weak interaction is similar to, but greater than, the EM interaction ! It is the massive W-boson in the propagator which makes it appear weak. For $q^2 \gg m_W^2$ weak interactions are more likely than EM.

Summary

★ Weak interaction is of form **Vector – Axial-vector (V-A)**

$$\frac{-ig_W}{\sqrt{2}}\frac{1}{2}\gamma^{\mu}(1-\gamma^5)$$

★ Consequently only **left-handed chiral particle states** and **right-handed chiral anti-particle states** participate in the weak interaction

➡ **MAXIMAL PARITY VIOLATION**

★ Weak interaction also violates Charge Conjugation symmetry

★ At low q^2 weak interaction is only weak because of the large W-boson mass

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2}$$

★ Intrinsic strength of weak interaction is similar to that of QED